

COHEN-MACAULAY APPROXIMATION IN FIBRED CATEGORIES

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ABSTRACT. We extend the Auslander-Buchweitz axioms and prove Cohen-Macaulay approximation results for fibred categories. Then we show that these axioms apply for the fibred category of pairs consisting of a finite type flat family of Cohen-Macaulay rings and modules. In particular such a pair admits an approximation with a flat family of maximal Cohen-Macaulay modules and a hull with a flat family of modules with finite injective dimension. The existence of minimal approximations and hulls in the local, flat case implies extension of upper semi-continuous invariants. As an example of MCM approximation we define a relative version of Auslander's fundamental module.

In the second part we study the induced maps of deformation functors and deduce properties like injectivity, isomorphism and smoothness under general, mainly cohomological conditions on the module. We also provide deformation theory for pairs (algebra, module), e.g. a cohomology for such pairs, a long exact sequence linking this cohomology to the André-Quillen cohomology of the algebra and the Ext cohomology of the module, Kodaira-Spencer classes and maps including a secondary Kodaira-Spencer class, and existence of a versal family for pairs with isolated singularity.

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1. INTRODUCTION

Let A be a Cohen-Macaulay ring of finite Krull dimension with a canonical module ω_A . Let MCM_A and FID_A denote the categories of maximal Cohen-Macaulay modules and of finite modules with finite injective dimension, respectively. M. Auslander and R.-O. Buchweitz proved in [6] that for any finite A -module N there exists short exact sequences

$$(1.0.1) \quad 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N \rightarrow L' \rightarrow M' \rightarrow 0$$

with M and M' in MCM_A and L and L' in FID_A . The maps $M \rightarrow N$ and $N \rightarrow L'$ in (1.0.1) are called a maximal Cohen-Macaulay approximation and a hull of finite injective dimension, respectively, of the module N . The association $N \mapsto X$ for X equal to M, M', L and L' define maps (functors) of corresponding stable categories. In this article we study the continuous properties of these maps.

Linear representations provided by (sheaves) of modules and the associated homological algebra plays an important role in algebra and algebraic geometry, e.g. as a means for classification by providing invariants. Finite complexes have particular properties as seen in the Buchsbaum-Eisenbud acyclicity criterion and the intersection theorems of Peskine, Szpiro and Roberts. However, for a non-regular local ring A , the standard homological invariants are given by the (generally) infinite minimal A -free resolutions, of which very little is known. To stay within finite complexes one can enlarge or change the category of resolving objects and Cohen-Macaulay approximation is a structured way of doing this.

Let D_A denote the subcategory $\text{Add}\{\omega_A\}$ of modules D isomorphic to direct summands of the $\omega_A^{\oplus r}$, $r > 0$. A part of the approximation result says that all the modules in FID_A have finite resolutions by objects in D_A . In particular the MCM approximation in (1.0.1) can be extended to a finite resolution

$$(1.0.2) \quad 0 \rightarrow D^{-n} \rightarrow D^{-n+1} \rightarrow \dots \rightarrow D^{-1} \rightarrow M \rightarrow N \rightarrow 0$$

with the D^i in D_A . In the case A is Gorenstein, D_A equals the category of finite projective modules P_A . This generalises: By a result of R. Y. Sharp [40] the functor $\text{Hom}_A(\omega_A, -)$ gives an exact equivalence $\text{D}_A \simeq \text{P}_A$, hence a finite projective resolution is associated to N . In the case A is local, the approximations and the complex can be chosen to be minimal and unique (with $D^i \cong \omega_A^{\oplus d^i}$) and in particular the d^i are invariants of N .

The developments since Auslander and Buchweitz' fundamental work [6] has included studies of invariants defined by Cohen-Macaulay approximation; [15, 8, 24] among several, 'injectivity' and 'surjectivity' properties of the approximation maps; [30, 44, 31], and characterisations of quasi-homogeneous isolated singularities; cf. [26, 34], all exclusively in the Gorenstein case. Noteworthy is [41] where A.-M. Simon and J. R. Strooker related some of these invariants with Hochster's Canonical Element Conjecture and the Monomial Conjecture. In particular these conjectures are equivalent to the vanishing of the δ -invariant of certain cyclic modules over all Gorenstein rings. S. P. Dutta applied the existence of a FID hull to prove a relationship between two of the Serre conjectures on intersection numbers: Failure of vanishing implies failure of higher non-negativity in the Gorenstein case under certain conditions, see [16].

Buchweitz' unpublished manuscript [12], an important precursor to [6], contains homological ideas which have become important in subsequent developments (e.g. [33]). Auslander and I. Reiten elaborated in [9] on [6], mainly with a view towards artin algebras, instigating several generalisations and analogies to Cohen-Macaulay approximation.

However, the ‘relative’ and continuous aspects have received surprisingly little attention. In [23] M. Hashimoto gave several new examples of Cohen-Macaulay approximation. Perhaps closest to our results is [23, IV 1.4.12] where an affine algebraic group G acts on a positively graded Cohen-Macaulay ring T which is flat over a regular base ring R . Hashimoto considers graded maximal Cohen-Macaulay T -modules (which automatically are R -flat) and graded modules locally of finite injective dimension (not R -flat in general), all with G -action. His result (with trivial group) is hence different from our Theorem 5.1. We also note some explicit 1-parameter families of indecomposable finite length modules N_t (for many Gorenstein rings) such that the minimal MCM approximation M_t is without free summands, see [43].

A central part of the classification problem is to prove the existence of objects with certain properties and to estimate ‘how many’ such objects there are. A natural question is thus whether there is Cohen-Macaulay approximation for flat families of modules. In Theorem 5.1 we give a positive answer to this question. For a Cohen-Macaulay (CM) map $h : S \rightarrow T$ (flat!) and an S -flat and finite T -module \mathcal{N} there are short exact sequences of S -flat and finite T -modules

$$(1.0.3) \quad 0 \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{N} \rightarrow \mathcal{L}' \rightarrow \mathcal{M}' \rightarrow 0$$

such that the fibres of these sequences give ‘absolute’ approximations and hulls as in the two sequences (1.0.1). Note that T in general is *not* a Cohen-Macaulay ring although the fibres of h are. We consider a category \mathbf{mod}^{fl} of pairs $\xi = (h : S \rightarrow T, \mathcal{N})$ and subcategories MCM, FID and D. They are fibred over the category CM of CM maps ($\xi \mapsto h$) and also fibred over the base category of noetherian rings ($\xi \mapsto S$). The approximation and the hull (1.0.3) induces functors of certain quotient categories fibred in additive categories over CM

$$(1.0.4) \quad \mathbf{mod}^{\text{fl}}/\text{D} \rightarrow \text{MCM}/\text{D} \quad \text{and} \quad \mathbf{mod}^{\text{fl}}/\text{D} \rightarrow \text{FID}/\text{D}$$

with analogous properties to the absolute case. If $h : S \rightarrow T$ is a *local* CM map, there is an analogous approximation result with minimal choices of the two sequences in (1.0.3), see Corollary 6.3.

A major consequence of these results is that any (numerical and additive) upper semi-continuous invariant of MCM or FID modules by the minimal approximations and hulls induces upper semi-continuous invariants for all finite modules, see Theorem 6.5. Examples of such invariants are given by the ω_A -ranks in the minimal *representing complex* $D^*(N)$ which is an (infinite) extension to the right of the D_A -complex in (1.0.2).

Auslander’s fundamental module E_A for a normal 2-dimensional singularity $\text{Spec } A$ is given by the CM approximation of the maximal ideal;

$$(1.0.5) \quad 0 \rightarrow \omega_A \rightarrow E_A \rightarrow \mathfrak{m}_A \rightarrow 0$$

which in a certain sense generates all almost split sequences for A , see [7]. As a general example of flat Cohen-Macaulay approximation we define the fundamental module for any finite type CM map of pure relative dimension ≥ 2 , see Corollary 7.3, and more generally a ‘fundamental’ functor of projective modules in Proposition 7.2.

An attractive feature of Auslander and Buchweitz’ theory is its axiomatic formulation with several applications besides the classical case described in the first paragraph, e.g. coherent rings with a cotilting module, the graded case, approximation with modules of Gorenstein dimension 0, and coherent sheaves on a projectively embedded Cohen-Macaulay scheme. See [6] and [23] for more examples. We formulate a relative Cohen-Macaulay approximation theory axiomatically in terms of categories $\text{D} \subseteq \text{X} \subseteq \text{A}$ fibred in abelian and additive subcategories over a base

category \mathbf{C} . In addition to the Auslander-Buchweitz axioms (AB1-4) for the fibre categories we formulate two axioms (BC1-2) regarding base change properties of the fibred categories. AB1-2 and BC1-2 imply the existence of an approximation and a hull which are preserved by any base change, see Theorem 4.4. If AB3 holds too, we get functoriality and adjointness properties in suitable stable categories fibred in additive categories, see Theorem 4.5. In the case described above $\mathbf{C} = \mathbf{CM}$, \mathbf{A} is the category \mathbf{mod} of pairs $(h : S \rightarrow T, \mathcal{N})$ where \mathcal{N} is a finite T -module (no S -flatness!) and $\mathbf{X} = \mathbf{MCM}$. Presently we consider only this example (and local variants of it), but there are several other natural applications of the relative theory.

In the second half of the article we proceed to study properties of continuous families of MCM approximations and FID hulls by homological methods. As a consequence of the existence of minimal approximations and hulls of local flat families there are induced natural maps of deformation functors of pairs of algebra and module:

$$(1.0.6) \quad \mathrm{Def}_{(A,N)} \longrightarrow \mathrm{Def}_{(A,X)} \quad \text{for } X = M, M', L \text{ and } L',$$

and there are corresponding maps $\mathrm{Def}_N^A \rightarrow \mathrm{Def}_X^A$ of deformation functors of the modules where A only deforms trivially. Rather weak grade conditions on N imply the injectivity of these maps for $X = L'$ (or L). If there in addition exists a versal family in $\mathrm{Def}_{(A,N)}$ (or Def_N^A) then the maps are isomorphisms for the appropriate category of henselian rings, see Theorem 9.4 and Corollary 9.5. As a consequence each CM algebraic k -algebra A with $A/\mathfrak{m}_A \cong k$ and $\dim A \geq 2$ has a finite A -module Q' of finite *projective* dimension with a *universal* deformation in $\mathrm{Def}_{Q'}^A(A)$, see Corollary 9.7. Proposition 9.8 says that if A is Gorenstein and $\dim A - \mathrm{depth} N = 1$ then $\mathrm{Def}_{(A,N)} \rightarrow \mathrm{Def}_{(A,M)}$ is smooth.

Consider a quotient ring $B = A/I$ defined by a regular sequence $I = (f_1, \dots, f_n)$ and an MCM B -module N . Then N is also an A -module with an MCM approximation $M \rightarrow N$. If N has a *lifting* to A/I^2 , then the composition of natural maps $\mathrm{Def}_N^B \rightarrow \mathrm{Def}_N^A \rightarrow \mathrm{Def}_M^A$ is injective, see Theorem 11.1. It turns out that the lifting condition is equivalent to the splitting of $B \otimes_A M \rightarrow N$. This generalises [8, 4.5] and might have a certain independent interest.

The second part of the article also contains some general deformation theory of a pair $(h : S \rightarrow T, \mathcal{N})$ of an algebra and a T -module. We define the graded algebra $\Gamma := T \oplus \mathcal{N}$ and consider the graded André-Quillen cohomology ${}_0\mathrm{H}^*(S, \Gamma, J)$ which govern the obstruction theory of the pair. In the case the graded Γ -module J is concentrated in degree 0 and 1 there is a natural long-exact sequence which in the case $J = \Gamma$ (with ${}_0\mathrm{H}^*(S, \Gamma) = {}_0\mathrm{H}^*(S, \Gamma, \Gamma)$) gives the suggestive

$$\begin{aligned} 0 \rightarrow \mathrm{End}_T(\mathcal{N}) \rightarrow {}_0\mathrm{Der}_S(\Gamma) \rightarrow \mathrm{Der}_S(T) \rightarrow \mathrm{Ext}_T^1(\mathcal{N}, \mathcal{N}) \rightarrow {}_0\mathrm{H}^1(S, \Gamma) \rightarrow \mathrm{H}^1(S, T) \\ \rightarrow \mathrm{Ext}_T^2(\mathcal{N}, \mathcal{N}) \rightarrow {}_0\mathrm{H}^2(S, \Gamma) \rightarrow \mathrm{H}^2(S, T) \rightarrow \dots \end{aligned}$$

It relates the cohomology of the pair with the cohomology groups governing the obstruction theory of the algebra T and of the module \mathcal{N} . The sequence is used in the proof of the existence of a versal element in $\mathrm{Def}_{(A,N)}$ where $\mathrm{Spec} A$ is an isolated equidimensional singularity and N is locally free on the smooth locus, see Theorem 10.1. It is also used to define and study the Kodaira-Spencer class $\kappa(\Gamma/S/\mathcal{O})$ in ${}_0\mathrm{H}^1(S, \Gamma, \Omega_{S/\mathcal{O}} \otimes \Gamma)$ (where $\mathcal{O} \rightarrow S$ is another ring homomorphism) which maps to the ungraded Kodaira-Spencer class $\kappa(T/S/\mathcal{O})$. In the case the latter is zero we define a ‘secondary’ Kodaira-Spencer class $\kappa(\sigma, \mathcal{N})$ in $\mathrm{Ext}_T^1(\mathcal{N}, \Omega_{S/\mathcal{O}} \otimes \mathcal{N})$ which depends on a choice of an S -algebra splitting σ . This enables us to define ‘global’ Kodaira-Spencer maps

$$(1.0.7) \quad g^\Gamma : \mathrm{Der}_{\mathcal{O}}(S) \rightarrow {}_0\mathrm{H}^1(S, \Gamma, \Gamma) \quad \text{and} \quad g^{(\sigma, \mathcal{N})} : \mathrm{Der}_{\mathcal{O}}(S) \rightarrow \mathrm{Ext}_T^1(\mathcal{N}, \mathcal{N}).$$

We also describe how classes and maps are related to the Atiyah class $\text{at}_{T/\mathcal{O}}(\mathcal{N})$ in $\text{Ext}_T^1(\mathcal{N}, \Omega_{T/\mathcal{O}} \otimes \mathcal{N})$. These results might have a certain independent interest. The arguments are general and can be formulated in the setting of L. Illusie's [28].

Injectivity of the corresponding local Kodaira-Spencer maps gives a criterion for a global flat family to be non-trivial. The Kodaira-Spencer maps commute with Cohen-Macaulay approximation and this is applied to show that injectivity of the Kodaira-Spencer map is preserved by Cohen-Macaulay approximation under certain 'global' conditions akin to the local ones in Theorem 9.4 and 11.1.

To make the text more reader friendly we have included some background material, e.g. on Cohen-Macaulay approximation and Kodaira-Spencer maps, and some of the central technical tools such as a general 'cohomology and base change' result and some language of fibred categories. Many results have analogous parts with similar arguments and the policy has been to give a fairly detailed proof of one case and leave the other cases to the reader.

2. PRELIMINARIES

All rings are commutative. If A is a ring, Mod_A denotes the category of A -modules and mod_A denotes the full subcategory of finite A -modules. If A is local then \mathfrak{m}_A denotes the maximal ideal. Subcategories are usually full and essential.

2.1. Categorical Cohen-Macaulay approximation. We briefly recall some of the main features of Cohen-Macaulay approximation as introduced by Auslander and Buchweitz in [6]. In this section let \mathbf{A} be an abelian category and $\mathbf{D} \subseteq \mathbf{X} \subseteq \mathbf{A}$ additive subcategories. Let $\hat{\mathbf{X}}$ denote the subcategory of \mathbf{A} of objects N which have finite resolutions $0 \rightarrow M_n \rightarrow \dots \rightarrow M_0 \rightarrow N \rightarrow 0$ with the M_i in \mathbf{X} . If n is the smallest such number, then $\mathbf{X}\text{-res.dim } N = n$. Let $\mathbf{X}\text{-inj.dim } N$ be the minimal n (possibly ∞) such that $\text{Ext}_{\mathbf{A}}^i(M, N) = 0$ for all $i > n$ and all M in \mathbf{X} . Let \mathbf{X}^\perp denote the subcategory of objects L in \mathbf{A} with $\mathbf{X}\text{-inj.dim } L = 0$; the right complement of \mathbf{X} . The left complement ${}^\perp\mathbf{X}$ is defined analogously.

Let N be an object in \mathbf{A} . An \mathbf{X} -approximation and a $\hat{\mathbf{D}}$ -hull of N are exact sequences as in (1.0.1) with L, L' in $\hat{\mathbf{D}}$ and M, M' in \mathbf{X} .

In general any $f : M \rightarrow N$ in \mathbf{A} is called a *right \mathbf{X} -approximation* of N if M is in \mathbf{X} and any $f' : M' \rightarrow N$ with M' in \mathbf{X} factorises through f . Dually, $g : N \rightarrow L$ is called a *left \mathbf{X} -approximation* of N if L is in \mathbf{X} and any $g' : N \rightarrow L'$ with L' in \mathbf{X} factorises through g .

Consider the following conditions on the triple of categories $(\mathbf{A}, \mathbf{X}, \mathbf{D})$.

- (AB1) \mathbf{X} is exact in \mathbf{A} (\mathbf{X} is closed under direct summands and extensions).
- (AB2) \mathbf{D} is a *cogenerator* for \mathbf{X} , i.e. for each object M in \mathbf{X} there is an object D in \mathbf{D} and a short exact sequence $M \rightarrow D \rightarrow M'$ with M' in \mathbf{X} .
- (AB3) \mathbf{D} is \mathbf{X} -injective, i.e. $\mathbf{D} \subseteq \mathbf{X}^\perp$.
- (AB4) \mathbf{A} -epimorphisms in \mathbf{X} are admissible (i.e. their kernels are contained in \mathbf{X}).

If AB1 and AB2, there exist \mathbf{X} -approximations and $\hat{\mathbf{D}}$ -hulls for all objects in $\hat{\mathbf{X}}$ [6, 1.1]. Assume AB1-3. Then any \mathbf{X} -approximation is a right \mathbf{X} -approximation and any $\hat{\mathbf{D}}$ -hull is a left $\hat{\mathbf{D}}$ -approximation. An \mathbf{X} -approximation determines a $\hat{\mathbf{D}}$ -hull and vice versa through the following diagram of short exact sequences; the upper horizontal and right vertical being an \mathbf{X} -approximation and a $\hat{\mathbf{D}}$ -hull of N , D is in

D. The boxed square is (co)cartesian (see [6, 1.4]):

$$(2.0.8) \quad \begin{array}{ccccc} L & \longrightarrow & M & \longrightarrow & N \\ \parallel & & \downarrow & \square & \downarrow \\ L & \longrightarrow & D & \longrightarrow & L' \\ & & \downarrow & & \downarrow \\ & & M' & \xlongequal{\quad} & M' \end{array}$$

Moreover, the category \mathbf{D} is determined by $\mathbf{X} \subset \mathbf{A}$. Indeed $\mathbf{D} = \mathbf{X} \cap \mathbf{X}^\perp$. By [6, 3.9] monomorphisms in $\hat{\mathbf{D}}$ are admissible and $\hat{\mathbf{D}} = \hat{\mathbf{X}} \cap \mathbf{X}^\perp$. Also $\mathbf{X} = {}^\perp \hat{\mathbf{D}} \cap \hat{\mathbf{X}} = {}^\perp \mathbf{D} \cap \hat{\mathbf{X}}$. If \mathbf{X}/\mathbf{D} denotes the quotient category, the \mathbf{X} -approximation induces a right adjoint to the inclusion functor $\mathbf{X}/\mathbf{D} \subseteq \hat{\mathbf{X}}/\mathbf{D}$ and the $\hat{\mathbf{D}}$ -hull induces a left adjoint to the inclusion functor $\hat{\mathbf{D}}/\mathbf{D} \subseteq \hat{\mathbf{X}}/\mathbf{D}$, see [6, 2.8].

A morphism $f : M \rightarrow N$ in \mathbf{A} is called *right minimal* if for any $g : M \rightarrow M$ with $fg = f$ it follows that g is an automorphism. Dually, f is called *left minimal* if for any $h : N \rightarrow N$ with $hf = f$ it follows that h is an automorphism. Note that if $f : M \rightarrow N$ and $f : M' \rightarrow N$ both are right minimal then there exists an isomorphism $g : M \rightarrow M'$ with $f = f'g$, and similarly for left minimal morphisms.

We will simply call an \mathbf{X} -approximation (a $\hat{\mathbf{D}}$ -hull) for minimal if it is right (left) minimal.

Example 2.1. Suppose A is a Cohen-Macaulay ring which possesses a canonical module ω_A in the sense that any localisation in a maximal ideal gives a maximal Cohen-Macaulay module of finite injective dimension and Cohen-Macaulay type 1, cf. [11, 3.3.16]. Let \mathbf{MCM}_A denote the category of maximal Cohen-Macaulay (MCM) A -modules and put $\mathbf{D}_A := \text{Add}\{\omega_A\}$. Then the triple $(\mathbf{A}, \mathbf{X}, \mathbf{D}) = (\mathbf{mod}_A, \mathbf{MCM}_A, \mathbf{D}_A)$ satisfies properties AB1-4, cf. [23, I 4.10.11] and $\hat{\mathbf{X}} = \mathbf{mod}_A$. If A in addition is a local ring, then the \mathbf{MCM}_A -approximation and the $\hat{\mathbf{D}}_A$ -hull can be chosen to be minimal, cf. [41, Sec. 3].

Let \mathbf{FID}_A^l denote the subcategory of finite A -modules E which have locally finite injective dimension, i.e. $\text{inj.dim}_{A_{\mathfrak{p}}} E_{\mathfrak{p}} < \infty$ for all $\mathfrak{p} \in \text{Spec } A$. The approximation result implies that $\mathbf{FID}_A^l = \hat{\mathbf{D}}_A$: Let L be in $\hat{\mathbf{D}}_A$. By induction on \mathbf{D}_A -res.dim L L is in \mathbf{FID}_A^l . Conversely let E be in \mathbf{FID}_A^l . If $L \rightarrow M \rightarrow E$ is an \mathbf{MCM}_A -approximation of E then M also has locally finite injective dimension. Let M^\vee denote $\text{Hom}_A(M, \omega_A)$ and choose a surjection $A^{\oplus n} \rightarrow M^\vee$. Both M^\vee and the kernel M_1 are MCM. Applying $\text{Hom}_A(-, \omega_A)$ gives (by duality theory) the short exact sequence $M \xrightarrow{i} \omega_A^{\oplus n} \rightarrow M_1^\vee$. But i splits since $\text{Ext}_A^1(M_1^\vee, M) = 0$ by [11, 3.3.3] and so M is in \mathbf{D}_A and E is in $\hat{\mathbf{D}}_A$.

2.2. The representing complex. Consider an abelian category \mathbf{A} and additive subcategories $\mathbf{D} \subseteq \mathbf{X} \subseteq \mathbf{A}$. A *DX-resolution* of an object N in \mathbf{A} is a finite resolution ${}^-C^* \twoheadrightarrow N$ with ${}^-C^i \in \mathbf{D}$ for $i < 0$ and ${}^-C^0 \in \mathbf{X}$. If $L := \text{coker}(d^{-2} : {}^-C^{-2} \rightarrow {}^-C^{-1})$, then the short exact sequence $L \rightarrow {}^-C^0 \rightarrow N$ is an \mathbf{X} -approximation. A *$\hat{\mathbf{D}}$ D-coresolution* of N is a coresolution $N \hookrightarrow {}^+C^*(N)$ such that ${}^+C^0 \in \hat{\mathbf{D}}$, ${}^+C^i \in \mathbf{D}$ and $\ker d^i \in \mathbf{X}$ for $i > 0$. If $M' := \ker d^1$ then the short exact sequence $N \rightarrow {}^+C^0 \rightarrow M'$ is a $\hat{\mathbf{D}}$ -hull. Given AB1 and AB2, each N in $\hat{\mathbf{X}}$ has a DX-resolution and a $\hat{\mathbf{D}}$ D-coresolution. Finally, a bounded below \mathbf{D} -complex $D^*(N) : \dots \rightarrow D^{-1} \rightarrow D^0 \rightarrow D^1 \rightarrow \dots$ with $\ker d^i \in \mathbf{X}$ for all $i \geq 0$ and its only non-trivial cohomology in degree zero with $H^0(D^*) \cong N$ is called a *D-complex representing N* . A representing complex splits into (and is reconstructed from) a DX-resolution given by $\dots \rightarrow D^{-1} \rightarrow \ker d^0 \twoheadrightarrow H^0(D^*) = N$ and a $\hat{\mathbf{D}}$ D-coresolution $N \hookrightarrow \text{coker } d^{-1} \rightarrow D^1 \rightarrow \dots$ where $N \hookrightarrow \text{coker } d^{-1}$ is induced by $\ker d^0 \hookrightarrow D^0$.

Lemma 2.2. Assume $\text{Ext}_A^1(\mathbf{X}, \hat{\mathbf{D}}) = 0$. Suppose $f : N_1 \rightarrow N_2$ is in $\hat{\mathbf{X}}$. Assume $F^*(N_i)$ exists for $i = 1, 2$ where $F^*(N_i)$ denotes one of the complexes $-C^*(N_i)$, $+C^*(N_i)$ or $D^*(N_i)$. Then f can be extended to an arrow of chain complexes $f^* : F^*(N_1) \rightarrow F^*(N_2)$ which is uniquely defined up to homotopy.

Assume AB1-3 for the triple of categories $(\mathbf{A}, \mathbf{X}, \mathbf{D})$. Then $N \mapsto -C^*(N)$, $N \mapsto +C^*(N)$ and $N \mapsto D^*(N)$ induce functors to the homotopy categories of chain complexes as follows:

$$-C^* : \hat{\mathbf{X}} \rightarrow \mathbf{K}^b(\mathbf{X}) \quad +C^* : \hat{\mathbf{X}} \rightarrow \mathbf{K}^+(\hat{\mathbf{D}}) \quad D^* : \hat{\mathbf{X}}/\mathbf{D} \rightarrow \mathbf{K}^+(\mathbf{D})$$

Proof. The proof for $-C^*(N)$ and $+C^*(N)$ follows standard lines for constructing chain maps and homotopies. The assumption $\text{Ext}_A^1(\mathbf{X}, \hat{\mathbf{D}}) = 0$ is used every time a lifting or extension of an arrow is required.

Let $(D_i^*, d_i^*) = D^*(N_i)$ and let $M_i = \ker d_i^0$ and $L_i = \text{im } d_i^{-1}$. Then there are short exact sequences $L_i \rightarrow M_i \rightarrow N_i$ which by assumption are \mathbf{X} -approximations. Since $\text{Ext}_A^1(M_1, L_2) = 0$, the arrow $N_1 \rightarrow N_2$ extends to the \mathbf{X} -approximation and further on to the negative part of the complexes. If $M'_i = \ker d_i^1$ then the M'_i are in \mathbf{X} by assumption and there are short exact sequences $M_i \rightarrow D_i^0 \rightarrow M'_i$. There is an extension of $M_1 \rightarrow D_2^0$ to $D_1^0 \rightarrow D_2^0$ and an induced arrow $M'_1 \rightarrow M'_2$ which again extends and so on to a chain map $f^* : D_1^* \rightarrow D_2^*$.

Let $g^* : D_1^* \rightarrow D_2^*$ be a chain map, put $g = H^0(g^*)$, $s = f - g$ and $s^* = f^* - g^*$. Suppose s factors through D in \mathbf{D} ; $s = ab$ with $a : D \rightarrow N_2$. Since $\text{Ext}_A^1(D, L_2) = 0$ there exist a lifting $\tilde{a} : D \rightarrow M_2$ of a . Put $h_N = \tilde{a}b$ and continue similarly to construct a homotopy h for the extended negative part:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & D_1^{-1} & \longrightarrow & M_1 & \longrightarrow & N_1 \longrightarrow 0 \\ & \searrow h^{-1} & \downarrow s^{-1} & \swarrow h_M & \downarrow Z^0(s^*) & \swarrow h_N & \downarrow s \\ \cdots & \longrightarrow & D_2^{-2} & \longrightarrow & D_2^{-1} & \longrightarrow & M_2 \longrightarrow N_2 \longrightarrow 0 \end{array}$$

In particular $h_M : M_1 \rightarrow D_2^{-1}$ can be extended to an $h^0 : D_1^0 \rightarrow D_2^{-1}$ with $s^{-1} = h^0 d_1^{-1} + d_2^{-2} h^{-1}$. The construction of the h^i for $i > 0$ is standard. \square

Lemma 2.3. Assume AB1-3 for the triple of categories $(\mathbf{A}, \mathbf{X}, \mathbf{D})$. Given an exact sequence $\varepsilon : 0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$ with objects in $\hat{\mathbf{X}}$. Then there are exact sequences of complexes where ε equals the cohomology:

- (i) $0 \rightarrow -C^*(N_1) \rightarrow -C^*(N_2) \rightarrow -C^*(N_3) \rightarrow 0$
- (ii) $0 \rightarrow +C^*(N_1) \rightarrow +C^*(N_2) \rightarrow +C^*(N_3) \rightarrow 0$
- (iii) $0 \rightarrow D^*(N_1) \rightarrow D^*(N_2) \rightarrow D^*(N_3) \rightarrow 0$ (termwise split exact)

Proof. Choose \mathbf{X} -approximations $L_i \rightarrow M_i \rightarrow N_i$ for $i = 1, 3$. There is an 3×3 commutative diagram of 6 short exact sequences which extends the “horseshoe” diagram, cf. [23, 1.12.11]. One obtains an \mathbf{X} -approximation of N_2 and short exact sequences $m : M_1 \rightarrow M_2 \rightarrow M_3$ and $L_1 \rightarrow L_2 \rightarrow L_3$ in \mathbf{X} and $\hat{\mathbf{D}}$ respectively since both categories are closed by extensions (by AB1 and [6, 3.8]). If $D_i^*[1] \xrightarrow{\eta_i} L_i$ are finite \mathbf{D} -resolutions then since $\text{Ext}_A^1(D_3^{-1}, L_1) = 0$ there is a lifting $\tilde{\eta}_3 : D_3^{-1} \rightarrow L_2$ of η_3 which combined with η_1 gives $\eta_2 : D_1^{-1} \coprod D_3^{-1} \rightarrow L_2$. The kernels of the resulting arrows between short exact sequences give a short exact sequence of objects in $\hat{\mathbf{D}}(S)$. The argument is repeated. Splicing with m in degree zero the short exact sequence of $-C^*$ -resolutions in (i) is obtained.

Choose short exact sequences $M_i \rightarrow D_i^0 \rightarrow M'_i$ for $i = 1, 3$ as in AB2. Since $\text{Ext}_A^1(M_3, D_1^0) = 0$ there is an extension to an arrow of short exact sequences from m to $D_1^0 \rightarrow D_2^0 \rightarrow D_3^0$ with $D_2^0 = D_1^0 \coprod D_3^0$ and $M'_2 := \text{coker}(M_2 \rightarrow D_2^0) \in \mathbf{X}$ by AB1. Repeated application of this argument gives a short exact sequence of

D-coresolutions and splicing with the sequences in (i) gives (iii). Pushout of $M_i \rightarrow D_i^0 \rightarrow M_i'$ along $M_i \rightarrow N_i$ gives a short exact sequence of \hat{D} -hulls and splicing with $D_i^1 \rightarrow D_i^2 \rightarrow \dots$ gives (ii). \square

2.3. Base change. The main tool for reducing properties to the fibres in a flat family will be the base change theorem. We follow the quite elementary and general approach of A. Ogus and G. Bergman [37].

Definition 2.4. Let $h : S \rightarrow T$ be a ring homomorphism, I an S -module, N a T -module. For each $u \in I$ there is a natural T -linear map $\varphi(u) : N \rightarrow N \otimes_S I$; $n \mapsto n \otimes u$. Let F be an S -linear functor of some additive subcategory of \mathbf{Mod}_S to \mathbf{Mod}_T . Then the *exchange map* e_I for F is defined as the T -linear map $e_I : F(S) \otimes_S I \rightarrow F(I)$ given by $\xi \otimes u \mapsto \varphi(u)_* \xi$. Let $\mathbf{m}\text{-Spec } T$ denote the set of closed points in $\text{Spec } T$.

Proposition 2.5. Let $h : S \rightarrow T$ be a ring homomorphism with S noetherian. Suppose $\{F^q : \mathbf{mod}_S \rightarrow \mathbf{mod}_T\}_{q \geq 0}$ is an h -linear cohomological δ -functor.

- (i) If the exchange map $e_{S/\mathfrak{n}}^q : F^q(S) \otimes_S S/\mathfrak{n} \rightarrow F^q(S/\mathfrak{n})$ is surjective for all \mathfrak{n} in $Z = \text{im}\{\mathbf{m}\text{-Spec } T \rightarrow \text{Spec } S\}$, then $e_I^q : F^q(S) \otimes_S I \rightarrow F^q(I)$ is an isomorphism for all I in \mathbf{mod}_S .
- (ii) If $e_{S/\mathfrak{n}}^{q+1}$ is surjective for all \mathfrak{n} in Z , then e_I^q is an isomorphism for all I in \mathbf{mod}_S if and only if $F^{q+1}(S)$ is S -flat.

Note that if the F^q in addition extend to functors of all S -modules $F^q : \mathbf{Mod}_S \rightarrow \mathbf{Mod}_T$ which commute with direct limits, then the conclusions are valid for all I in \mathbf{Mod}_S .

Example 2.6. Suppose S and T are noetherian. Let $K^* : K^0 \rightarrow K^1 \rightarrow \dots$ be a complex of S -flat and finite T -modules. Define $F^q : \mathbf{mod}_S \rightarrow \mathbf{mod}_T$ by $F^q(I) = H^q(K^* \otimes_S I)$. Then $\{F^q\}_{q \geq 0}$ is an h -linear cohomological δ -functor which extends to all S -modules and commutes with direct limits.

Example 2.7. Suppose S and T are noetherian. Let M and N be finite T -modules with N S -flat. Then the functors $F^q : \mathbf{mod}_S \rightarrow \mathbf{mod}_T$ defined by $F^q(I) = \text{Ext}_T^q(M, N \otimes_S I)$ for $q \geq 0$ give an h -linear cohomological δ -functor which extends to all S -modules and commutes with direct limits.

Let $S \rightarrow T$ and $S \rightarrow S'$ be ring homomorphisms, M a T -module, $T' = T \otimes_S S'$ and N' a T' -module. Then there is a change of rings spectral sequence

$$(2.7.1) \quad E_2^{p,q} = \text{Ext}_{T'}^q(\text{Tor}_p^S(M, S'), N') \Rightarrow \text{Ext}_{T'}^{p+q}(M, N')$$

which, in addition to the isomorphism $\text{Hom}_{T'}(M \otimes_S S', N') \cong \text{Hom}_T(M, N')$, gives edge maps $\text{Ext}_{T'}^q(M \otimes_S S', N') \rightarrow \text{Ext}_T^q(M, N')$ for $q > 0$ which are isomorphisms too if M (or S') is S -flat. If I' is an S' -module we can compose the exchange map $e_{I'}^q$ (regarding I' as S -module) with the inverse of this edge map for $N' = N \otimes_S I'$ and obtain a map $c_{I'}^q$ of T' -modules

$$(2.7.2) \quad c_{I'}^q : \text{Ext}_T^q(M, N) \otimes_S I' \rightarrow \text{Ext}_{T'}^q(M \otimes_S S', N \otimes_S I').$$

Remark 2.8. This is the base change map (in the affine case) considered by A. Altman and S. Kleiman, their conditions are slightly different, see [1, 1.9].

We will use the following geometric notation. Suppose $h : S \rightarrow T$ is a ring homomorphism, M is a T -module and s is a point in $\text{Spec } S$ with residue field $k(s)$. Then M_s denotes the fibre $M \otimes_S k(s)$ of M at s with its natural $T_s = T \otimes_S k(s)$ -module structure. Now Proposition 2.5 implies the following:

Corollary 2.9. *Suppose $S \rightarrow T$ and $S \rightarrow S'$ are homomorphisms of noetherian rings, M and N are finite T -modules, $Z = \text{im}(\mathfrak{m}\text{-Spec } T \rightarrow \text{Spec } S)$ and q is an integer. Assume that M (if $q > 0$) and N are S -flat.*

- (i) *If $\text{Ext}_{T_s}^{q+1}(M_s, N_s) = 0$ for all s in Z , then c_I^q in (2.7.2) is an isomorphism for all S' -modules I .*
- (ii) *If in addition $\text{Ext}_{T_s}^{q-1}(M_s, N_s) = 0$ for all $s \in Z$, then $\text{Ext}_T^q(M, N)$ is S -flat.*

3. CATEGORIES FIBRED IN ADDITIVE CATEGORIES

We will phrase our results in the language of fibred categories.¹ We therefore briefly recall some of the basic notions, taken mainly from A. Vistoli's article in [18]. Then we define quotients of categories fibred in additive categories.

Consider a category \mathbf{C} . Given a category over \mathbf{C} , i.e. a functor $p : \mathbf{F} \rightarrow \mathbf{C}$. To an object T in \mathbf{C} , let $\mathbf{F}(T)$; the *fiber of \mathbf{F} over T* , denote the subcategory of arrows φ in \mathbf{F} such that $p(\varphi) = \text{id}_T$. An arrow $\varphi_1 : \xi \rightarrow \xi_1$ in \mathbf{F} is *cocartesian* if for any arrow $\varphi_2 : \xi \rightarrow \xi_2$ in \mathbf{F} and any arrow $f_{21} : p(\xi_1) \rightarrow p(\xi_2)$ in \mathbf{C} with $f_{21}p(\varphi_1) = p(\varphi_2)$ there exists a unique arrow $\varphi_{21} : \xi_1 \rightarrow \xi_2$ with $p(\varphi_{21}) = f_{21}$ and $\varphi_{21}\varphi_1 = \varphi_2$. If for any arrow $f : T \rightarrow T'$ in \mathbf{C} and any object ξ in \mathbf{F} with $p(\xi) = T$ there exists a cocartesian arrow $\varphi : \xi \rightarrow \xi'$ for some ξ' with $p(\varphi) = f$, then \mathbf{F} (or rather $p : \mathbf{F} \rightarrow \mathbf{C}$) is a *fibred category*. Moreover, ξ' will be called a *base change* of ξ by f . If ξ'' is another base change of ξ by f then ξ' and ξ'' are isomorphic over T' by a unique isomorphism. We shall also say that a property P of objects in the fibres of \mathbf{F} is *preserved by base change* if $P(\xi)$ implies $P(\xi')$ for any base change ξ' of ξ . A morphism of fibred categories is a functor $F : \mathbf{F}_1 \rightarrow \mathbf{F}_2$ with $p_2 F = p_1$ such that φ cocartesian implies $F(\varphi)$ cocartesian. If F in addition is an inclusion of categories, \mathbf{F}_1 is a fibred subcategory of \mathbf{F}_2 . A category with all arrows being isomorphisms is a groupoid. A fibred category \mathbf{F} over \mathbf{C} is called a *category fibred in groupoids* (often abbreviated to groupoid) if all fibres $\mathbf{F}(T)$ are groupoids. Then all arrows in \mathbf{F} are cocartesian. If all fibres $\mathbf{F}(T)$ only contain identities, then \mathbf{F} is called a *category fibred in sets*.

Lemma 3.1. *Given functors $F : \mathbf{F} \rightarrow \mathbf{G}$ and $q : \mathbf{G} \rightarrow \mathbf{C}$ and suppose q is fibred in sets. Then F is fibred (in groupoids/sets) if and only if qF is fibred (in groupoids/sets).*

If T is an object in a category \mathbf{C} let \mathbf{C}/T denote the comma category of arrows to T . Then the forgetful functor $\mathbf{C}/T \rightarrow \mathbf{C}$ is fibred in sets. If $p : \mathbf{F} \rightarrow \mathbf{C}$ is fibred (in groupoids/sets), ξ is an object in \mathbf{F} and $T = p(\xi)$, then there is a natural functor $p_\xi : \mathbf{F}/\xi \rightarrow \mathbf{C}/T$. The composition $\mathbf{F}/\xi \rightarrow \mathbf{F} \rightarrow \mathbf{C}$ is clearly fibred (in groupoids/sets) and hence $\mathbf{F}/\xi \rightarrow \mathbf{C}/T$ is fibred (in groupoids/sets) by Lemma 3.1. If $p : \mathbf{F} \rightarrow \mathbf{C}$ is a functor and \mathbf{C}' is a subcategory of \mathbf{C} we can define the *restriction* $p' : \mathbf{F}_{|\mathbf{C}'} \rightarrow \mathbf{C}'$ of \mathbf{F} to \mathbf{C}' by picking for $\mathbf{F}_{|\mathbf{C}'}$ the objects and morphisms in \mathbf{F} that p takes into \mathbf{C}' . It follows that $\mathbf{F}_{|\mathbf{C}'}$ is fibred (in groupoids/sets) if \mathbf{F} is.

The composition of two cocartesian arrows is cocartesian and isomorphisms are cocartesian. Hence the subcategory \mathbf{F}_{coCa} of cocartesian arrows in a fibred category \mathbf{F} over \mathbf{C} is fibred in groupoids. If \mathbf{F} is fibred in groupoids there is an associated category fibred in sets $\bar{\mathbf{F}} \rightarrow \mathbf{C}$ defined by identifying all isomorphic objects in all fibres $\mathbf{F}(T)$ and identifying arrows accordingly. If \mathbf{F} is fibred in sets one defines a functor $F : \mathbf{C} \rightarrow \mathbf{Sets}$ by $F(T) := \mathbf{F}(T)$ and $F(f) : F(T) \rightarrow F(T')$ is defined by $F(f)(\xi) := \eta_{\xi, f}$ where $\varphi_{\xi, f} : \xi \rightarrow \eta_{\xi, f}$ is the (in this case) unique cocartesian lifting

¹We have chosen to work with rings instead of (affine) schemes. Our definition of a fibred category $p : \mathbf{F} \rightarrow \mathbf{C}$ reflects this choice and is equivalent to the functor of opposite categories $p^{\text{op}} : \mathbf{F}^{\text{op}} \rightarrow \mathbf{C}^{\text{op}}$ being a fibred category as defined in [18]. M. Artin called p a cofibred category in [3], but this seems not to be common usage.

of f . From a functor $G : \mathbf{C} \rightarrow \mathbf{Sets}$ one defines a category fibred in sets, and these two operations are inverse up to natural equivalences.

Definition 3.2. An *additive (abelian) category F over \mathbf{C}* is a functor $p : F \rightarrow \mathbf{C}$ such that:

- (i) The fibre $F(T)$ is an additive (abelian) category for all objects T in \mathbf{C} .
- (ii) For all objects ξ_1 and ξ_2 in F and arrows $f : p(\xi_1) \rightarrow p(\xi_2)$ in \mathbf{C} ,

$$\mathrm{Hom}_f(\xi_1, \xi_2) := \{\varphi \in \mathrm{Hom}_F(\xi_1, \xi_2) \mid p(\varphi) = f\}$$

is an abelian group, and composition of arrows

$$\mathrm{Hom}_{f_2}(\xi_2, \xi_3) \times \mathrm{Hom}_{f_1}(\xi_1, \xi_2) \rightarrow \mathrm{Hom}_{f_2 f_1}(\xi_1, \xi_3)$$

is bilinear.

A morphism $F : F_1 \rightarrow F_2$ of additive (abelian) categories over \mathbf{C} is a *linear* functor F over \mathbf{C} , i.e. which gives linear maps of Hom-groups. If in addition F is an inclusion of categories then F_1 is an additive (abelian) subcategory of F_2 over \mathbf{C} . A category F over \mathbf{C} is *fibred in additive (abelian) categories*, abbreviated by FAd (FAB), if F is both fibred and additive (abelian) over \mathbf{C} . Morphisms should be linear and preserve cocartesian arrows. A FAd subcategory is a morphism of FAds which is an inclusion of categories. For $i = 1, 2$ let A_i be a FAb over \mathbf{C} and $X_i \subseteq A_i$ a FAd subcategory such that the fibre categories $X_i(T)$ are exact. Then a morphism of FAds $F : X_1 \rightarrow X_2$ is *exact* if F preserves short exact sequences for all the fibre categories.

Note that in a FAd finite (co)products in the fibres are preserved by base change. Given a FAd subcategory $D \subseteq F$. Two arrows φ_1 and φ_2 in F are D -equivalent if $p(\varphi_1) = p(\varphi_2)$ and $\varphi_1 - \varphi_2$ factors through an object in D . Write $\varphi_1 \sim \varphi_2$. Define the quotient category F/D over \mathbf{C} to have the same objects as F and $\mathrm{Hom}_{F/D}(\xi_1, \xi_2) := \mathrm{Hom}_F(\xi_1, \xi_2) / \sim$. The natural map to \mathbf{C} makes F/D an additive category over \mathbf{C} and the natural functor $F \rightarrow F/D$ is linear over \mathbf{C} .

Lemma 3.3. *If $\varphi_1 : \xi \rightarrow \xi_1$ is cocartesian in F and $\varphi : \xi_1 \rightarrow \xi_2$ is any arrow such that $\varphi\varphi_1 \sim 0$ then $\varphi \sim 0$.*

Proof. Suppose $\varphi\varphi_1 = \beta\alpha$ with $\alpha : \xi \rightarrow \delta$ and with δ in D . If $p(\beta) : T' \rightarrow T_2$ then since D is a fibred subcategory there exists an arrow $\delta \rightarrow \delta_2$ which is cocartesian in F and with $p(\delta_2) = p(\xi_2) = T_2$. Replacing δ with δ_2 we assume $p(\delta) = T_2$. Since φ_1 is cocartesian there exists a unique arrow $\tau : \xi_1 \rightarrow \delta$ with $\tau\varphi_1 = \alpha$. Since φ_1 is cocartesian uniqueness implies that $\beta\tau = \varphi$. \square

Lemma 3.4. *Given a FAd subcategory $D \subseteq F$ over \mathbf{C} , then the quotient category F/D is FAd over \mathbf{C} and the quotient morphism $F \rightarrow F/D$ is a morphism of FAds.*

Proof. We first show that if $\varphi_1 : \xi \rightarrow \xi_1$ is cocartesian in F then its image $[\varphi_1]$ in F/D is cocartesian. Given $\varphi_2 : \xi \rightarrow \xi_2$ and $\theta : \xi_1 \rightarrow \xi_2$ with $\theta\varphi_1 = \varphi_2$. Suppose $\theta' : \xi_1 \rightarrow \xi_2$ with $p(\theta') = p(\theta)$ satisfies $\theta'\varphi_1 \sim \varphi_2$. If $\varphi = \theta' - \theta$ then $\varphi\varphi_1 \sim 0$ so by Lemma 3.3 $\varphi \sim 0$. Now we show that $[\theta]$ is independent of the representations of the other maps. Let $\varphi'_i : \xi \rightarrow \xi_i$ with $\varphi'_i \sim \varphi_i$ and suppose (as we may) that θ' satisfies $\theta'\varphi'_1 = \varphi'_2$ with $p(\theta') = p(\theta)$. Again let $\varphi = \theta' - \theta$. Then $0 \sim \varphi'_2 - \varphi_2 = \theta'\varphi'_1 - \theta\varphi_1 = \theta'(\varphi'_1 - \varphi_1) + \varphi\varphi_1 \sim \varphi\varphi_1$. By Lemma 3.3 $\varphi \sim 0$. Given $f : T \rightarrow T_1$ and ξ in F/D with $p(\xi) = T$ there exists a cocartesian $\varphi_1 : \xi \rightarrow \xi_1$ in F with $p(\varphi_1) = f$ and by what we have done $[\varphi_1]$ is cocartesian in F/D . \square

Note that there are in general more cocartesian arrows in F/D than those in the image of cocartesian arrows in F . The following lemma characterises the cocartesian arrows in the quotient category:

Lemma 3.5. *If ρ and θ are composeable arrows in \mathbf{F} with ρ cocartesian and θ inducing an isomorphism in \mathbf{F}/\mathbf{D} , then $[\theta\rho]$ is cocartesian in \mathbf{F}/\mathbf{D} . Conversely, suppose $[\varphi] : \xi_1 \rightarrow \xi_2$ is cocartesian in \mathbf{F}/\mathbf{D} over $f : T_1 \rightarrow T_2$. Then for any base change $\rho : \xi_1 \rightarrow \xi_1^\#$ of ξ_1 over f in \mathbf{F} , the induced arrow $\varphi^\# : \xi_1^\# \rightarrow \xi_2$ gives an isomorphism in $\mathbf{F}/\mathbf{D}(T_2)$.*

Proof. If $\rho : \xi_1 \rightarrow \xi_2$ is cocartesian and $[\theta] : \xi_2 \rightarrow \xi_3$ is an isomorphism, let $\varphi = \theta\rho$. If $\tau : \xi_1 \rightarrow \xi_4$, $p(\xi_i) = T_i$ and there is a map $f : T_3 \rightarrow T_4$ with $p(\tau) = fp(\theta)p(\rho)$, then there is a unique arrow $\mu : \xi_2 \rightarrow \xi_4$ above $fp(\theta)$ with $\mu\rho = \tau$. This gives the arrow $[\mu][\theta]^{-1} : \xi_3 \rightarrow \xi_4$. If $\mu_i : \xi_3 \rightarrow \xi_4$ for $i = 1, 2$ are two arrows with $[\mu_i][\varphi] = [\tau]$, then $[\mu_1][\theta] = [\mu_2][\theta]$ since $[\rho]$ is cocartesian in \mathbf{F}/\mathbf{D} by Lemma 3.4. Since $[\theta]$ is an isomorphism, $[\mu_1] = [\mu_2]$.

Conversely, since $[\varphi]$ is cocartesian there is a unique arrow $[\psi] : \xi_2 \rightarrow \xi_1^\#$ in $\mathbf{F}/\mathbf{D}(T_2)$ with $[\psi\varphi] = [\rho]$. By Lemma 3.4 $[\rho]$ is cocartesian. It follows that $[\varphi^\#] = [\psi]^{-1}$. \square

4. COHEN-MACAULAY APPROXIMATION IN FIBRED CATEGORIES

Given a category \mathbf{C} and a category \mathbf{A} fibred in abelian categories over \mathbf{C} . Base change by an $f : T \rightarrow T'$ in \mathbf{C} applied to the objects in a complex $\dots \rightarrow N_d \rightarrow N_{d-1} \rightarrow \dots$ in $\mathbf{A}(T)$ can by Lemma 3.3 be uniquely extended to a complex and yield a commutative diagram where the vertical arrows are the cocartesian base change arrows:

$$\begin{array}{ccccccc} \dots & \longrightarrow & N_{d+1} & \longrightarrow & N_d & \longrightarrow & N_{d-1} \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & N_{d+1}^\# & \longrightarrow & N_d^\# & \longrightarrow & N_{d-1}^\# \longrightarrow \dots \end{array}$$

Similarly base change of a commutative diagram Δ in $\mathbf{A}(T)$ gives a commutative diagram $\Delta^\#$ and the base change arrows give an arrow of diagrams $\Delta \rightarrow \Delta^\#$.

Let $\mathbf{X} \subseteq \mathbf{A}$ be a FAd subcategory. Consider the following two conditions on the pair (\mathbf{A}, \mathbf{X}) and an object T in \mathbf{C} .

- (BC1) If $\alpha : A_1 \rightarrow A_2$ is an epimorphism in $\mathbf{A}(T)$ and $f : T \rightarrow T'$ is an arrow in \mathbf{C} then any base change of α by f is an epimorphism in $\mathbf{A}(T')$.
- (BC2) Let $\xi : 0 \rightarrow A \rightarrow B \rightarrow M \rightarrow 0$ be an exact sequence in $\mathbf{A}(T)$ with M in $\mathbf{X}(T)$ and $f : T \rightarrow T'$ is an arrow in \mathbf{C} . Then any base change of ξ by f is an exact sequence in $\mathbf{A}(T')$.

The first condition would be satisfied if base change had a right adjoint. The second condition mimics flatness for all objects in $\mathbf{X}(T)$.

The following is an elementary, but essential technical consequence of BC1.

Lemma 4.1. *Let \mathbf{A} be a category fibred in abelian categories over \mathbf{C} which satisfies BC1 for T in \mathbf{C} . Let $c : \dots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \dots$ be an acyclic complex in $\mathbf{A}(T)$ which remains exact after a base change $\dots \rightarrow L_n^\# \rightarrow L_{n-1}^\# \rightarrow \dots$ of c by $f : T \rightarrow T'$. Then base change of $K_n := \ker\{d_{n-1} : L_{n-1} \rightarrow L_{n-2}\}$ by f is isomorphic to $\ker d_{n-1}^\#$ for all n .*

Proof. Let $Q_n = \ker d_{n-1}^\#$. Since the composition $K_n^\# \rightarrow L_{n-1}^\# \rightarrow L_{n-2}^\#$ by Lemma 3.3 is zero (as $K_n \rightarrow L_n^\#$ is cocartesian), there is a factorisation $\rho : K_n^\# \rightarrow Q_n$ of $K_n^\# \rightarrow L_{n-1}^\#$. On the other hand the composition $L_{n+1}^\# \rightarrow L_n^\# \rightarrow K_n^\#$ is zero too, hence there is an arrow from $\text{coker } d_{n+1}^\# \cong Q_n$ to $K_n^\#$ which is a section of ρ . By assumption $L_n^\# \rightarrow K_n^\#$ is an epimorphism. It follows that $Q_n \cong K_n^\#$. \square

Definition 4.2. Given FAd subcategories $D \subseteq X \subseteq A$. Let $\hat{X}^{\text{fl}}(T)$ denote the additive subcategory of $A(T)$ with objects N which have a finite X -resolution $M_* \rightarrow N$ which is preserved as resolution by any base change. Let $\hat{X}^{\text{fl}} \subseteq A$ denote the resulting FAd subcategory. Let $\hat{D}^{\text{fl}}(T)$ denote the additive subcategory of $A(T)$ with objects L which have a $D(T)$ -resolution $D^* \rightarrow L$ which is preserved as resolution by any base change. Let $\hat{D}^{\text{fl}} \subseteq A$ denote the resulting FAd subcategory.

The reasoning in the beginning of this section combined with Lemma 2.2 gives the following.

Lemma 4.3. *Let $\eta : \dots \rightarrow E_n \rightarrow E_{n-1} \rightarrow \dots$ and $\lambda : \dots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots$ be complexes in $A(T)$ and $\eta^{\#}$ and $\lambda^{\#}$ the complexes resulting from base change over $f : T \rightarrow T'$. If η is homotopic to λ then $\eta^{\#}$ is homotopic to $\lambda^{\#}$.*

In particular; if N in $A(T)$ has one DX-resolution ($\hat{D}D$ -coresolution) which is preserved by base change then all DX-resolutions ($\hat{D}D$ -coresolutions) are preserved by base change.

Theorem 4.4. *Let A be a category fibred in abelian categories over C and let $D \subseteq X \subseteq A$ be inclusion morphisms of categories fibred in additive categories. Fix an object T in C . Assume BC1-2 for (A, X) and T , and AB1-2 for the triple of categories $(A(T), X(T), D(T))$. Then any object N in $\hat{X}^{\text{fl}}(T)$ admits an $X(T)$ -approximation and a $\hat{D}(T)$ -hull;*

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N \rightarrow L' \rightarrow M' \rightarrow 0$$

with M and M' in $X(T)$ and L and L' in $\hat{D}^{\text{fl}}(T)$, which are preserved by any base change.

Proof. The proof is a variation of the original proof of [6, 1.1]. For every N in $\hat{X}^{\text{fl}}(T)$ let $r(N)$ denote the minimal length of an $X(T)$ -resolution $M_* \rightarrow N$ which is preserved by base change. The proof is by induction on $r(N)$. If $r(N) = 0$ then N is in X and so is its own X -approximation, while AB2 provides a short exact sequence $N \rightarrow D \rightarrow M'$ which is a $\hat{D}(T)$ -hull with D in $D(T) \subseteq \hat{D}^{\text{fl}}(T)$. The approximation is trivially preserved by base change, the hull because of BC2. Assume $r = r(N) > 0$ and let $0 \rightarrow M_r \rightarrow \dots \rightarrow M_0 \rightarrow N$ be an $X(T)$ -resolution of minimal length preserved by base change. Then $N_1 = \ker(M_0 \rightarrow N)$ is in $\hat{X}^{\text{fl}}(T)$ by Lemma 4.1 and $r(N_1) = r - 1$. By induction there is a $\hat{D}(T)$ -hull $N_1 \rightarrow L \rightarrow M'_1$ with L in \hat{D}^{fl} which is preserved by base change. Pushout of $e : N_1 \rightarrow M_0 \rightarrow N$ along $N_1 \rightarrow L$ gives an $X(T)$ -approximation $L \rightarrow M \rightarrow N$ by AB1. In the commutative diagram obtained by a base change;

$$(4.4.1) \quad \begin{array}{ccccc} N_1^{\#} & \longrightarrow & M_0^{\#} & \longrightarrow & N^{\#} \\ \downarrow & & \downarrow & & \parallel \\ L^{\#} & \longrightarrow & M^{\#} & \longrightarrow & N^{\#} \\ \downarrow & & \downarrow & & \\ (M'_1)^{\#} & \equiv & (M'_1)^{\#} & & \end{array}$$

the upper row (by Lemma 4.1) and the columns (by BC2) are short exact sequences. It follows that the middle row is a short exact sequence.

By AB2 there is a short exact sequence $M \rightarrow D \rightarrow M'$ with D in $D(T)$ and M' in $X(T)$. Pushout of $M \rightarrow D \rightarrow M'$ along $M \rightarrow N$ gives a short exact sequence $h : N \rightarrow L' \rightarrow M'$. Since the induced sequence $L \rightarrow D \rightarrow L'$ is short exact, L' is contained in $\hat{D}(T)$. Applying a base change we obtain the following commutative

diagram:

$$(4.4.2) \quad \begin{array}{ccccc} L^\# & \longrightarrow & M^\# & \longrightarrow & N^\# \\ \parallel & & \downarrow & & \downarrow \\ L^\# & \longrightarrow & D^\# & \longrightarrow & (L')^\# \\ & & \downarrow & & \downarrow \\ & & (M')^\# & = & (M')^\# \end{array}$$

The upper row and (by BC2) the two columns are short exact sequences. It follows that the middle row is a short exact sequence and hence that L' is contained in $\hat{D}^\#$. \square

Sequences as in Theorem 4.4 preserved by any base change will be called an X -approximation and a $\hat{D}^\#$ -hull of N respectively.

Lemma 3.4 makes the following definition reasonable. Three categories fibred in additive categories (FAds) A_i , $i = 1, 2, 3$, an inclusion of FAds $A_1 \subseteq A_2$, and a morphism of FAds $F : A_2 \rightarrow A_3$ equivalent to the quotient morphism $A_2 \rightarrow A_2/A_1$ is called a *short exact sequence of categories fibred in additive categories* and is denoted by $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$.

Theorem 4.5. *Let A be a category fibred in abelian categories over C and let $D \subseteq X \subseteq A$ be inclusion morphisms of categories fibred in additive categories. Assume BC1-2 for the pair (A, X) and AB1-3 for the triple of categories $(A(T), X(T), D(T))$, for all objects T in C . Then:*

- (i) *The X -approximation induces a morphism of categories fibred in additive categories $j^! : \hat{X}^\# / D \rightarrow X / D$ which is a right adjoint to the full and faithful inclusion morphism $j_! : X / D \rightarrow \hat{X}^\# / D$.*
- (ii) *The $\hat{D}^\#$ -hull induces a morphism of categories fibred in additive categories $i^* : \hat{X}^\# / D \rightarrow \hat{D}^\# / D$ which is a left adjoint to the full and faithful inclusion morphism $i_* : \hat{D}^\# / D \rightarrow \hat{X}^\# / D$.*
- (iii) *Together these maps give the following commutative diagram of short exact sequences of categories fibred in additive categories:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \hat{D}^\# / D & \xrightarrow{i_*} & \hat{X}^\# / D & \xrightarrow{j^!} & X / D \longrightarrow 0 \\ & & \text{id} \downarrow & & \parallel & & \uparrow \text{id} \\ 0 & \longleftarrow & \hat{D}^\# / D & \xleftarrow{i^*} & \hat{X}^\# / D & \xleftarrow{j_!} & X / D \longleftarrow 0 \end{array}$$

Proof. In each fibre most of these statements are true by the arguments in the proof of [6, 2.8] since we have Theorem 4.4. The general cases are reduced to fibre cases by applying base change. First we have to establish the functors. Note that the quotient categories involved are FAds over C by Lemma 3.4. Let $p : A \rightarrow C$ denote the fibration. For each N_i in $\hat{X}^\#$ put $T_i = p(N_i)$ and choose a $\hat{D}^\#$ -hull $\iota_i : N_i \rightarrow L_i \rightarrow M_i$ which exists by Theorem 4.4 and such that $\iota_i = \text{id}$ if N_i is in $\hat{D}^\#$. For each arrow $\psi : N_1 \rightarrow N_2$ choose an arrow $\lambda_{21} : L_1 \rightarrow L_2$ commuting with ψ . This arrow is obtained as a composition of a base change $L_1 \rightarrow L_1^\#$ over $p(\psi) : T_1 \rightarrow T_2$ with an extension $L_1^\# \rightarrow L_2$ of $N_1^\# \rightarrow L_2$ obtained since $\text{Ext}_{A(T_2)}^1(M_1^\#, L_2) = 0$ by [6, 2.5]. If composable it follows from [6, 2.8] that $\lambda_{32}\lambda_{21} \sim \lambda_{31}$.

There is a unique arrow $\varphi : N_1^\# \rightarrow N_2$ induced by ψ . If $\lambda'_{21} : L_1 \rightarrow L_2$ is an extension of $\psi' : N_1 \rightarrow N_2$ with $p(\psi') = p(\psi)$ such that $\delta_1 := \psi - \psi'$ is equivalent to 0, we have by Lemma 3.3 that $\delta : N_1^\# \rightarrow N_2$ induced from δ_1 by base change factors through an object D in $D(T_2)$. It follows that δ factors through $N_1^\# \rightarrow L_1^\#$.

Let τ denote the composition $L_1^\# \rightarrow N_2 \rightarrow L_2$ (so $\tau \sim 0$). Let η be a base change over $p(\psi)$ of the difference of the two extensions; $\eta = (\lambda_{21} - \lambda'_{21})^\#$. One calculates that $(\eta - \tau)\iota_1^\# = 0$, hence $\eta - \tau$ is induced by an arrow $M_1^\# \rightarrow L_2$ which lifts to an arrow $M_1^\# \rightarrow D^0$ where $D^* \twoheadrightarrow L_2$ is a finite D-resolution of L_2 (since $\text{Ext}_{\mathcal{A}(T_2)}^1(M_1^\#, \hat{D}(T_2)) = 0$). Hence $\eta - \tau \sim 0$, so $\eta \sim 0$ and $\lambda_{21} \sim \lambda'_{21}$. We have shown that $i^* : \hat{X}^\text{fl}/D \rightarrow \hat{D}^\text{fl}/D$ is a well defined functor. To show that i^* preserves cocartesian arrows we apply Lemma 3.5: If $[\psi] : N_1 \rightarrow N_2$ is cocartesian in \hat{X}^fl/D then the induced map $[\varphi] : N_1^\# \rightarrow N_2$ is an isomorphism and by [6, 2.8] so is any extension $L_1^\# \rightarrow L_2$ of $[\varphi]$. Composed with the base change $L_1 \rightarrow L_1^\#$ we get a cocartesian arrow in \hat{D}^fl/D by Lemma 3.5.

A similar argument gives that the morphism $j^! : \hat{X}^\text{fl}/D \rightarrow X/D$ induced by (choices of) X-approximation also is well defined as a map of fibred categories.

To prove adjointness for the pair $(j_!, j^!)$ consider the chosen X-approximation $L \rightarrow M \xrightarrow{\pi} N$ of N in $\hat{X}^\text{fl}(T)$. Given $\varphi_1 : M_1 \rightarrow N$ with M_1 in $X(T_1)$ and $f = p(\varphi_1)$. Let $\varphi : M_1^\# \rightarrow N$ be induced by a base change of M_1 by f . Since $\text{Ext}_{\mathcal{A}(T)}^1(M_1^\#, L) = 0$, φ can be lifted to an arrow $\psi : M_1^\# \rightarrow M$. Composing ψ with the base change $M_1 \rightarrow M_1^\#$ gives a lifting of φ_1 which shows surjectivity of the adjointness map $[\pi \circ -]$. To prove injectivity consider for $i = 2, 3$ arrows $\psi_i : M_1 \rightarrow M$ in X with $\pi\psi_2 = \pi\psi_3$. Since $p(\pi) = \text{id}$ we have $p(\psi_2) = p(\psi_3) = f$ and we can define $\psi_1 = \psi_2 - \psi_3$ with $\pi\psi_1 \sim 0$. Base change by f induces a $\psi : M_1^\# \rightarrow M$ from ψ_1 . Lemma 3.3 gives $\pi\psi \sim 0$. The argument in [6, 2.8] implies that ψ and hence ψ_1 factors through an object in $D(T)$. Analogous arguing gives the adjointness of the pair (i^*, i_*) .

The commutativity of the diagram in (iii) follows by definition. For $i^*j_! = 0 = j^!i_*$ see [6, 2.8]. We prove exactness in the upper row. Given $\varphi : N_1 \rightarrow N_2$ in \hat{X}^fl with $f = p(\varphi) : T_1 \rightarrow T_2$ such that $j^![\varphi] = 0$. If $\pi_i : M_i \rightarrow N_i$ are the chosen X-approximations, $j^![\varphi]$ is represented by a lifting $\psi : M_1 \rightarrow M_2$ and the assumption is that ψ factors through an object D of D . We claim that φ factors through an object in \hat{D}^fl . By base change it's sufficient to prove the special case $f = \text{id}_{T_2}$. If M is any object in $X(T)$ we have that the composition $\text{Ext}_{\mathcal{A}(T)}^1(M, N_1) \cong \text{Ext}_{\mathcal{A}(T)}^1(M, M_1) \xrightarrow{\psi_*} \text{Ext}_{\mathcal{A}(T)}^1(M, M_2) \cong \text{Ext}_{\mathcal{A}(T)}^1(M, N_2)$ is φ_* which hence equals 0. If $e : N_1 \rightarrow L'_1 \rightarrow M'_1$ is a \hat{D}^fl -hull of N_1 , the connecting takes φ to $\varphi_*e \in \text{Ext}_{\mathcal{A}(T)}^1(M'_1, N_2)$, i.e. to 0, and so there exists a $\delta : L'_1 \rightarrow N_2$ which induces φ . Exactness in the lower row is analogous. \square

Proposition 4.6. *Let \mathcal{A} be a category fibred in abelian categories over \mathcal{C} and let $D \subseteq X \subseteq \mathcal{A}$ be inclusion morphisms of categories fibred in additive categories. Fix an object T in \mathcal{C} . Assume BC1-2 for (\mathcal{A}, X) and T , and AB1-3 for the triple of categories $(\mathcal{A}(T), X(T), D(T))$. Then:*

- (i) $\hat{D}^\text{fl}(T) = \hat{X}^\text{fl}(T) \cap X(T)^\perp$ and $D(T) = X(T) \cap \hat{D}^\text{fl}(T)$.
- (ii) $\hat{X}^\text{fl}(T)$ and $\hat{D}^\text{fl}(T)$ are closed under extensions.
- (iii) Exact sequences $\dots \rightarrow N_n \xrightarrow{d_n} N_{n-1} \rightarrow \dots$ with objects N_i and kernels $\ker d_i$ in $\hat{X}^\text{fl}(T)$ remain exact after base change.

If in addition AB4, then:

- (iv) Epimorphisms in $\hat{X}^\text{fl}(T)$ are admissible.
- (v) $\text{Add } \hat{X}^\text{fl}(T) = \hat{X}^\text{fl}(T)$ and $\text{Add } \hat{D}^\text{fl}(T) = \hat{D}^\text{fl}(T)$.

Proof. For (i) the proofs of [6, 3.6-7] works with the fl-s too by Theorem 4.4. By (i) it's sufficient to prove (ii) for $\hat{X}^\text{fl}(T)$. Let $e : N_1 \rightarrow N_2 \rightarrow N_3$ be a short exact sequence in $\hat{X}(T)$. If N_1 and N_3 are in $\hat{X}^\text{fl}(T)$, Theorem 4.4 and Lemma

4.3 implies that $-C^*(N_1)$ and $-C^*(N_3)$ in Lemma 4.3 are preserved as resolutions by base change. Together with BC2 this implies that base change of the short exact sequence of resolutions in Lemma 2.3 (i) gives a short exact sequence of DX-resolutions. For (iii) the long exact sequence is broken into short exact sequences ξ_i with objects in $\hat{X}^{\text{fl}}(T)$. By Lemma 4.1 it is sufficient to prove the claim for short exact sequences. By Lemma 2.3 the short exact sequence of DX-resolutions in Lemma 2.3 (i) is preserved by base change. It follows that the short exact sequences ξ_i remain exact after base change.

For (iv); if N_2 and N_3 in e are in $\hat{X}^{\text{fl}}(T)$ then N_1 is in $\hat{X}(T)$ by AB4, see [6, 3.5]. The argument proceeds as for extensions.

By (i) it's sufficient to prove (v) for \hat{X}^{fl} . If $N_1 \amalg N_2$ is an object in $\hat{X}^{\text{fl}}(T)$, then N_i is in $\hat{X}(T)$ for $i = 1, 2$ by [6, 3.4]. By Lemma 4.3 $-C^*(N_1) \amalg -C^*(N_2)$ is preserved by base change as resolution of $N_1 \amalg N_2$. It follows that the resolution $-C^*(N_i)$ is preserved by base change for $i = 1, 2$. \square

Corollary 4.7. *Assume BC1-2 and AB1-3 as in Proposition 4.6. If N is in $\hat{X}^{\text{fl}}(T)$ then any DX-resolution $-C^*(N) \twoheadrightarrow N$ and any $\hat{\text{D}}\text{D}$ -coresolution $N \rightarrowtail +C^*(N)$ as in Section 2.2 is preserved by base change.*

Proof. By Theorem 4.4 there exist a DX-resolution and a $\hat{\text{D}}\text{D}$ -coresolution in \hat{X}^{fl} . The result follows from Proposition 4.6 (iii) and Lemma 4.3. \square

Definition 4.8. Let \mathbf{A} be a category fibred in abelian categories over \mathbf{C} and let $\mathbf{D} \subseteq \mathbf{A}$ be an inclusion morphism of a category fibred in additive categories. For T in \mathbf{C} let $\hat{\mathbf{D}}^{\text{fl}}(T)$ denote the full subcategory of $\mathbf{A}(T)$ of objects K with a finite coresolution $K \rightarrow L^*$ with objects L^i in $\hat{\mathbf{D}}^{\text{fl}}(T)$ for $i \geq 0$.

Lemma 4.9. *With these notions we have:*

- (i) *Epimorphisms in $\hat{\mathbf{D}}^{\text{fl}}(T)$ are admissible if and only if $\hat{\mathbf{D}}^{\text{fl}}(T) = \check{\mathbf{D}}^{\text{fl}}(T)$.*

Assume BC1-2 and AB1-4 for $(\mathbf{A}(T), \mathbf{X}(T), \mathbf{D}(T))$. Then:

- (ii) $\hat{\mathbf{D}}^{\text{fl}}(T) = \hat{\mathbf{D}}(T) \cap \check{\mathbf{D}}^{\text{fl}}(T)$.
- (iii) *Epimorphisms in $\hat{\mathbf{D}}^{\text{fl}}(T)$ are admissible if epimorphisms in $\hat{\mathbf{D}}(T)$ are admissible.*

Proof. (i) is trivially true. In (ii) $\hat{\mathbf{D}}^{\text{fl}}(T) \subseteq \hat{\mathbf{D}}(T) \cap \check{\mathbf{D}}^{\text{fl}}(T)$ is obvious. For the other inclusion, suppose K is an object in $\hat{\mathbf{D}}(T) \cap \check{\mathbf{D}}^{\text{fl}}(T) \setminus \hat{\mathbf{D}}^{\text{fl}}(T)$ with a $\hat{\mathbf{D}}^{\text{fl}}(T)$ -coresolution $K \rightarrow L^*$ of length $n > 0$. Since monomorphisms are admissible in $\hat{\mathbf{D}}(T) = \hat{\mathbf{X}}(T) \cap \mathbf{X}(T)^{\perp}$, all $K^i = \ker(L^i \rightarrow L^{i+1})$ are contained in $\hat{\mathbf{D}}(T)$, and we can assume $n = 1$. But then K has to be in $\hat{\mathbf{X}}^{\text{fl}} \cap \hat{\mathbf{D}} = \hat{\mathbf{D}}^{\text{fl}}(T)$ by Proposition 4.6. Since (i) is true “without the fl” by [6, 4.1], (iii) follows immediately from (i) and (ii). \square

5. COHEN-MACAULAY APPROXIMATION OF FLAT FAMILIES

We define fibred categories of Cohen-Macaulay maps with flat modules and show that they allow Cohen-Macaulay approximation in the finite type case and the local, algebraic case.

5.1. The finite type case. Let $h : S \rightarrow T$ be a ring homomorphism of noetherian rings. We say that h is a *Cohen-Macaulay (CM) map* if it is of finite type, faithfully flat and all fibres are Cohen-Macaulay (cf. [19, 6.8.1]). In particular h is equidimensional ([20, 15.4.1]). B. Conrad has defined the *dualising module* ω_h for any CM h , see [13, Sec. 3.5]. Suppose h has pure relative dimension n . For some $N \geq n$ there is a surjective S -algebra map $P \rightarrow T$ where $P = S[t_1, \dots, t_N]$. Let $\omega_{P/S} := \wedge^N \Omega_{P/S}$. Then there is an isomorphism $\omega_h \cong \text{Ext}_P^{N-n}(T, \omega_{P/S})$ which is

natural in the factorisation $S \rightarrow P \rightarrow T$, see [13, 3.5.3-6]. By (local) duality theory and Corollary 2.9 ω_h is S -flat (or see [13, Cor. 3.5.2]). If S is a field, we have that ω_h is a canonical module of T as in Example 2.1, cf. [11, 3.3.7 and 16].

Let \mathbf{CM} be the category with objects the CM maps and morphisms $(g, f) : h_1 \rightarrow h_2$ pairs of ring homomorphisms $g : S_1 \rightarrow S_2$ and $f : T_1 \rightarrow T_2$ such that $h_2 g = f h_1$ and such that the induced map $f \otimes 1 : T_1 \otimes S_2 \rightarrow T_2$ is an isomorphism:

$$\begin{array}{ccc} T_1 & \xrightarrow{f} & T_2 \xleftarrow{\simeq} T_1 \otimes_{S_1} S_2 \\ h_1 \uparrow & & \uparrow h_2 \\ S_1 & \xrightarrow{g} & S_2 \end{array}$$

Let \mathbf{R} denote the category of noetherian rings. The forgetful functor $p : \mathbf{CM} \rightarrow \mathbf{R}; (g, f) \mapsto g$, makes \mathbf{CM} fibred in groupoids over \mathbf{R} . The essential part is that \mathbf{CM} should allow base change, i.e. given $g : S_1 \rightarrow S_2$ and $h_1 : S_1 \rightarrow T_1$ as above there should exist a T_2 , an $h_2 : S_2 \rightarrow T_2$ and an f such that (g, f) is a morphism $h_1 \rightarrow h_2$ in \mathbf{CM} . This follows from [20, 15.4.3].

Let \mathbf{mod} be the category of pairs $(h : S \rightarrow T, N)$ with h in \mathbf{CM} and N a finite T -module. A morphism $(h_1, N_1) \rightarrow (h_2, N_2)$ is a morphism $(g, f) : h_1 \rightarrow h_2$ in \mathbf{CM} and a f -linear map $\alpha : N_1 \rightarrow N_2$. Then α is cocartesian with respect to the forgetful functor $F : \mathbf{mod} \rightarrow \mathbf{CM}$ if $1 \otimes \alpha : T_2 \otimes N_1 \rightarrow N_2$ is an isomorphism. It follows that \mathbf{mod} is fibred in abelian categories over \mathbf{CM} . Adding the property that N is S -flat gives the full subcategory \mathbf{mod}^{fl} . Moreover, let \mathbf{MCM} be the full subcategory of \mathbf{mod}^{fl} where the fibre $N_s = N \otimes_S k(s)$ is a maximal Cohen-Macaulay T_s -module for all $s \in \text{Spec } S$. The inclusions $\mathbf{MCM} \subseteq \mathbf{mod}^{\text{fl}} \subseteq \mathbf{mod}$ are inclusion morphisms of categories fibred in additive categories (FAdS) over \mathbf{CM} . For \mathbf{MCM} this follows from [20, 15.4.3]. If h is a CM map let $\mathbf{mod}_h, \mathbf{MCM}_h, \dots$ denote the fibre categories of $\mathbf{mod}, \mathbf{MCM}, \dots$ over h . An object in \mathbf{MCM}_h is called an $(h-)$ family of maximal Cohen-Macaulay modules (or shorter; an MCM h -module).

Given a morphism $h_1 \rightarrow h_2$ in \mathbf{CM} . By [13, Thm. 3.6.1] there is a natural isomorphism with base change $T_2 \otimes \omega_{h_1} \cong \omega_{h_2}$ which is compatible with localisation of T_1 and is functorial with respect to composition $h_1 \rightarrow h_2 \rightarrow h_3$. It follows that $h \mapsto (h, \omega_h)$ defines a morphism $\omega : \mathbf{CM} \rightarrow \mathbf{MCM}$ of fibred categories over \mathbf{R} which is a section of the forgetful $F : \mathbf{MCM} \rightarrow \mathbf{CM}$. Let \mathbf{D} be the full subcategory of \mathbf{MCM} over \mathbf{CM} with the objects (h, D) where D is an object in $\text{Add}\{\omega_h\}$. The inclusion $\mathbf{D} \subseteq \mathbf{MCM}$ is an inclusion of FAdS over \mathbf{CM} .

If \mathbf{U} denotes any of these FAdS over \mathbf{CM} , let $\underline{\mathbf{U}}$ denote the quotient ('stable') category \mathbf{U}/\mathbf{D} . With this notation we have the following.

Theorem 5.1. *The pair $(\mathbf{mod}, \mathbf{MCM})$ over \mathbf{CM} satisfies BC1-2 and the triple of fibre categories $(\mathbf{mod}_h, \mathbf{MCM}_h, \mathbf{D}_h)$ satisfies AB1-4 for all objects h in \mathbf{CM} . Moreover:*

- (i) *The fibred categories $\widehat{\mathbf{MCM}}^{\text{fl}}$ and $\hat{\mathbf{D}}^{\text{fl}}$ equals \mathbf{mod}^{fl} and $\hat{\mathbf{D}} \cap \mathbf{mod}^{\text{fl}}$ respectively.*
- (ii) *For any object (h, N) in \mathbf{mod}^{fl} , N admits an MCM-approximation and a $\hat{\mathbf{D}}^{\text{fl}}$ -hull which in particular are preserved by any base change.*
- (iii) *The MCM-approximation induces a morphism of categories fibred in additive categories $j^! : \underline{\mathbf{mod}}^{\text{fl}} \rightarrow \underline{\mathbf{MCM}}$ which is a right adjoint to the full and faithful inclusion morphism $j_! : \underline{\mathbf{MCM}} \rightarrow \underline{\mathbf{mod}}^{\text{fl}}$.*
- (iv) *The $\hat{\mathbf{D}}^{\text{fl}}$ -hull induces a morphism of categories fibred in additive categories $i^* : \underline{\mathbf{mod}}^{\text{fl}} \rightarrow \underline{\hat{\mathbf{D}}^{\text{fl}}}$ which is a left adjoint to the full and faithful inclusion morphism $i_* : \underline{\hat{\mathbf{D}}^{\text{fl}}} \rightarrow \underline{\mathbf{mod}}^{\text{fl}}$.*

- (v) Together these maps give the following commutative diagram of short exact sequences of categories fibred in additive categories:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \hat{\mathbf{D}}^{\text{fl}} & \xrightarrow{i_*} & \mathbf{mod}^{\text{fl}} & \xrightarrow{j^!} & \mathbf{MCM} \longrightarrow 0 \\
 & & \text{id} \downarrow & & \parallel & & \uparrow \text{id} \\
 0 & \longleftarrow & \hat{\mathbf{D}}^{\text{fl}} & \xleftarrow{i^*} & \mathbf{mod}^{\text{fl}} & \xleftarrow{j_!} & \mathbf{MCM} \longleftarrow 0
 \end{array}$$

Proof. Since base change is given by the tensor product BC1 and BC2 follows. In particular, a short exact sequence $e : N_1 \rightarrow N_2 \rightarrow N_3$ in \mathbf{mod}_h with N_3 and either N_1 or N_2 in \mathbf{MCM}_h gives short exact sequences of MCMs after base change to each fibre T_s , i.e. AB1 and AB4.

For AB2, suppose M is in \mathbf{MCM}_h . Then $M^\vee = \text{Hom}_T(M, \omega)$ is in \mathbf{MCM}_h too by Corollary 2.9. Since M^\vee is finite there is a short exact sequence $M^\vee \leftarrow T^r \leftarrow M_1$. By AB1 M_1 is in \mathbf{MCM}_h . Applying $\text{Hom}_T(-, \omega_h)$ gives the desired short exact sequence since Corollary 2.9 implies that $\text{Ext}_T^1(M^\vee, \omega_h) = 0$ and that the natural map $M \rightarrow M^{\vee\vee}$ is an isomorphism. AB3 also follows from Corollary 2.9.

Any N in $\widehat{\mathbf{MCM}}_h^{\text{fl}}$ has by definition a finite MCM-resolution (say of length n) preserved by base change. Since objects in \mathbf{MCM} are S -flat it follows by induction on n that $\text{Tor}_1^S(N, k(s)) = 0$ for all $s \in \text{Spec } S$. Hence N is S -flat. Conversely, if N is in $\mathbf{mod}_h^{\text{fl}}$ it follows that a sufficiently high syzygy of N is in \mathbf{MCM}_h , i.e. N is in $\widehat{\mathbf{MCM}}_h^{\text{fl}}$. With Proposition 4.6 this gives $\hat{\mathbf{D}}_h^{\text{fl}} = \mathbf{mod}_h^{\text{fl}} \cap \mathbf{MCM}_h^\perp$. By induction on the length of the resolution $\hat{\mathbf{D}}_h \subseteq \mathbf{MCM}_h^\perp$ and so $\mathbf{mod}_h^{\text{fl}} \cap \hat{\mathbf{D}}_h \subseteq \hat{\mathbf{D}}_h^{\text{fl}}$. The opposite inclusion is clear by the first part of (i). Now (ii)-(v) follows directly from Theorem 4.5. \square

Corollary 5.2. *Let $h : S \rightarrow T$ be an object in CM. Then:*

- (i) $\mathbf{D}_h = \mathbf{MCM}_h^\perp \cap \mathbf{MCM}_h$ and $\hat{\mathbf{D}}_h^{\text{fl}} = \mathbf{MCM}_h^\perp \cap \mathbf{mod}_h^{\text{fl}}$
- (ii) *The kernel of a surjective map in $\hat{\mathbf{D}}_h^{\text{fl}}$ is contained in $\hat{\mathbf{D}}_h^{\text{fl}}$.*

Proof. (ii): Note that if $N_1 \rightarrow N_2 \rightarrow N_3$ is a short exact sequence with N_2 and N_3 in $\hat{\mathbf{D}}_h^{\text{fl}}$, in particular S -flat, then N_1 has to be S -flat too. To show that N_1 is in $\hat{\mathbf{D}}_h$ we use the criterion in [6, 4.6] to show that $\check{\mathbf{D}}_h = \hat{\mathbf{D}}_h$: Suppose that M is in \mathbf{MCM}_h . Assume that M satisfies $\mathbf{MCM}_h\text{-inj.dim } M = n < \infty$ which by [6, 4.3] is equivalent to the existence of a coresolution of M of length n in \mathbf{D}_h . The fibre at any $s \in \text{Spec } S$ gives a \mathbf{D}_{T_s} -coresolution of the MCM T_s -module M_s . Since $\check{\mathbf{D}}_{T_s} = \hat{\mathbf{D}}_{T_s}$ ([6, 6.3]) it follows by [6, 4.6] that M_s is contained in \mathbf{D}_{T_s} . Since $\mathbf{D}_{T_s} = \mathbf{MCM}_{T_s} \cap \mathbf{MCM}_{T_s}^\perp$ by [6, 3.7] it follows from Corollary 2.9 that $\mathbf{MCM}_h\text{-inj.dim } M = 0$ and so M is in $\mathbf{MCM}_h \cap \mathbf{MCM}_h^\perp$. But by Theorem 5.1 we can invoke [6, 3.7] again which gives $\mathbf{MCM}_h \cap \mathbf{MCM}_h^\perp = \mathbf{D}_h$ so M is in \mathbf{D}_h . By [6, 4.6(d)] $\check{\mathbf{D}}_h = \hat{\mathbf{D}}_h$ follows and N_1 is in $\hat{\mathbf{D}}_h$. \square

Let \mathbf{P} denote the fibred subcategory of \mathbf{MCM} over \mathbf{CM} of pairs (h, P) with $h : S \rightarrow T$ in \mathbf{CM} and P a finite projective T -module. Let $\hat{\mathbf{P}}^{\text{fl}}$ denote the full subcategory of \mathbf{mod}^{fl} of pairs (h, Q) such that Q has a finite projective dimension. The inclusions of categories fibred in additive categories $\mathbf{P} \subseteq \hat{\mathbf{P}}^{\text{fl}} \subseteq \mathbf{mod}^{\text{fl}}$ are closed under extensions over \mathbf{CM} .

Lemma 5.3. *There is an exact equivalence $\hat{\mathbf{D}}^{\text{fl}} \simeq \hat{\mathbf{P}}^{\text{fl}}$ of categories fibred in additive categories defined by the functor $(h, L) \mapsto (h, \text{Hom}_T(\omega_h, L))$ with a quasi-inverse $(h, Q) \mapsto (h, Q \otimes_T \omega_h)$. It induces an equivalence of fibred quotient categories $\hat{\mathbf{D}}^{\text{fl}}/\mathbf{D} \simeq \hat{\mathbf{P}}^{\text{fl}}/\mathbf{P}$.*

Proof. By base change we reduce to the absolute case where the equivalence is well known, cf. [23, I 4.10.16]. The functor $\text{Hom}_-(\omega_-, -)$ takes a map $\varphi : L_1 \rightarrow L_2$ over $h_1 \rightarrow h_2$ to the map $\text{Hom}_{T_1}(\omega_{h_1}, L_1) \rightarrow \text{Hom}_{T_2}(\omega_{h_2}, L_2)$; $\alpha \mapsto (\varphi\alpha)^\#$, the unique map which pulls back to $\varphi\alpha$ by the cocartesian map $\omega_{h_1} \rightarrow \omega_{h_2}$. It is a well defined functor since the dualising module is functorial. If L is in \hat{D}_h^{fl} then $\text{Ext}_{T_s}^i(\omega_{T_s}, L_s) = 0$ for all $i \neq 0$ and $s \in \text{Spec } S$. By Corollary 2.9 the functor is exact and $\text{Hom}_T(\omega_h, L)$ is S -flat. In particular $\text{End}_T(\omega_h) \cong T$. If D is in D_h then $\text{Hom}_T(\omega_h, D)$ is projective as a direct summand of a free module. If $D^* \rightarrow L$ is a finite D_h -resolution of L , then $\text{Hom}_T(\omega_h, D^*)$ gives a projective resolution of $\text{Hom}_T(\omega_h, L)$ since $\hat{D}_h^{\text{fl}} \subseteq \text{MCM}_h^\perp$ (Corollary 5.2). The natural map $\text{ev} : \text{Hom}_T(\omega_h, L) \otimes \omega_h \rightarrow L$ commutes with base change to the fibres where it is an isomorphism ([11, 9.6.5]) and Nakayama's lemma and S -flatness implies that ev is an isomorphism too. Let $P^* \rightarrow Q$ be a P -resolution of Q in \hat{P}_h^{fl} of length n . Define covariant functors $G^i : \text{mod}_S \rightarrow \text{mod}_T$ by $G^i(V) := H^{i-n}(P^* \otimes_T \omega_h \otimes_S V)$. Since $P^* \otimes_T \omega_h$ is S -flat, $\{G^i\}$ defines a cohomological δ -functor. Since $P^* \otimes k(s)$ gives a resolution of $Q \otimes_S k(s)$ for all $s \in \text{Spec } S$, [11, 9.6.5] and Proposition 2.5 implies that $G^n(S) = Q \otimes_T \omega_h$ is S -flat and $P^* \otimes_T \omega_h \rightarrow Q \otimes_T \omega_h$ is a D -resolution. Moreover, the natural map $Q \rightarrow \text{Hom}_T(\omega_h, Q \otimes \omega_h)$ is an isomorphism by Nakayama's lemma again. \square

Example 5.4. Assume A is a Cohen-Macaulay ring with a canonical module ω_A . Given $L_i \in \hat{D}_A$ and put $Q_i = \text{Hom}_A(\omega_A, L_i)$ for $i = 1, 2$. If I is an injective resolution of L_2 and P is a projective resolution of Q_1 then both spectral sequences of $\text{Hom}_A(P, \text{Hom}_A(\omega_A, I))$ collapse at page 2 (use [23, I 4.10.19]) to give canonical isomorphisms

$$(5.4.1) \quad \text{Ext}_A^*(L_1, L_2) \cong \text{Ext}_A^*(Q_1, Q_2)$$

Remark 5.5. In his unpublished manuscript Buchweitz gave a construction of Cohen-Macaulay approximation for finite A -modules if A is a not necessarily commutative Gorenstein ring, see [12]. The MCM -approximation and the \hat{D}^{fl} -hull in Theorem 5.1 can be given essentially by the same construction. Let N be a finite T -module which is S -flat where $h : S \rightarrow T$ is a finite type Cohen-Macaulay map. Let $P = P(N) \rightarrow N$ be a projective resolution of N (i.e. by finite projective T -modules). Then $P^\vee = \text{Hom}_T(P, \omega_h)$ is a bounded below complex with bounded cohomology because $\text{Ext}_T^i(N, \omega_h) = 0$ for i greater than the relative dimension d of h by Corollary 2.9 ($\text{inj.dim } \omega_{T_s} = \dim T_s \leq d$). Then we can choose a projective resolution $f : P(P^\vee) \rightarrow P^\vee$ of P^\vee which is bounded above. Let $C = C(f)$ be the mapping cone of f . The modules in C are direct sums of projective modules and modules in D_h and the (co)kernels in the acyclic C are modules in MCM_h . By Corollary 2.9 it follows that C^\vee is acyclic too. There is a composition of natural maps $P \cong P^{\vee\vee} \rightarrow P(P^\vee)^\vee := G$ which hence is a quasi-isomorphism. But then G is just the representing complex of N and the MCM -approximation and the \hat{D} -hull is obtained as in Section 2.2. In the the case of coherent rings with a cotilting module (a concept introduced by Y. Miyashita, see [36, p. 142]) J.-i. Miyachi implicitly gave the same construction in [35, 3.2].

Suppose h has pure relative dimension and N_s is a non-zero Cohen-Macaulay T_s -module with $n = \dim T_s - \dim N_s$ constant for all $s \in \text{Spec } S$, i.e. N is a family of Cohen-Macaulay modules of codepth n . Put $N^\vee := \text{Ext}_T^n(N, \omega_h)$. Then $P^\vee = P(N)^\vee$ is quasi-isomorphic to $H^n(P^\vee) = N^\vee$ by duality theory and Corollary 2.9. So if $(F, d) \rightarrow N^\vee$ is a projective resolution of N^\vee , the representing complex is given as $G = F^\vee$. If $\text{Syz}_i^T N^\vee$ denotes the i^{th} syzygy module $\text{im } d_i$ and d_i^\vee denotes $\text{Hom}_T(d_i, \omega_h)$ then the commutative diagram (2.0.8) with short exact sequences is

given as

$$\begin{array}{ccccc}
 \mathrm{im}(d_n^\vee) & \longrightarrow & \mathrm{Hom}_T(\mathrm{Syz}_n^T N^\vee, \omega_h) & \longrightarrow & N^{\vee\vee} \\
 \parallel & & \downarrow & \square & \downarrow \\
 \mathrm{im}(d_n^\vee) & \longrightarrow & \mathrm{Hom}_T(F_n, \omega_h) & \longrightarrow & \mathrm{coker}(d_n^\vee) \\
 & & \downarrow & & \downarrow \\
 & & \mathrm{Hom}_T(\mathrm{Syz}_{n+1}^T N^\vee, \omega_h) & = & \mathrm{Hom}_T(\mathrm{Syz}_{n+1}^T N^\vee, \omega_h)
 \end{array}$$

with the MCM_h -approximation and \hat{D}_h -hull of N given by the upper horizontal and right vertical sequence, respectively, since $N^{\vee\vee} \cong N$ by duality theory and Corollary 2.9. Let C denote the mapping cone of a comparison map $P(N) \rightarrow G$. The homology of the truncated short exact sequence of complexes $G \rightarrow C \rightarrow P(N)[-1]$ gives the MCM_h -approximation $\mathrm{coker} d_{i+2}^G \rightarrow \mathrm{coker} d_{i+2}^C \rightarrow \mathrm{Syz}_i^T N$ for $i \geq 0$ (with $i = 0$ being the upper horizontal sequence in the diagram). In the absolute case (with N Cohen-Macaulay) this latter construction of the MCM approximation of $\mathrm{Syz}_i N$ was given by J. Herzog and A. Martsinkovsky in [26, 1.1].

5.2. Local cases.

We formulate local variants of the approximation theorem.

Fix a field k . Let \mathbf{H} denote the category of local, henselian, noetherian rings S with residue field $S/\mathfrak{m}_S \cong k$ and with local ring homomorphisms. A map $h : S \rightarrow T$ in \mathbf{H} is *algebraic* (or T is an algebraic S -algebra) if there is a finite type S -algebra \tilde{T} and a maximal ideal \mathfrak{m} in \tilde{T} with $\tilde{T}/\mathfrak{m} \cong k$ such that h is given by henselisation of \tilde{T} in \mathfrak{m} . Fix an algebraic k -algebra A which is supposed to be Cohen-Macaulay. Objects in the category \mathbf{hCM} are algebraic and flat S -algebras T with $T \otimes_S k \cong A$. A morphism $h_1 \rightarrow h_2$ is a pair of commuting maps $f : T_1 \rightarrow T_2$ and $g : S_1 \rightarrow S_2$ in \mathbf{H} as for the finite type case, giving a cocartesian square. Fibre sum exists in \mathbf{H} and is given by the henselisation of the tensor product $T = T_1 \otimes_{S_1} S_2$ in the maximal ideal $\mathfrak{m}_{T_1} T + \mathfrak{m}_{S_2} T$. We denote it by $T_1 \tilde{\otimes}_{S_1} S_2$. It has the same closed fibre as $S_1 \rightarrow T_1$ and it follows that the obvious functor $\mathbf{hCM} \rightarrow \mathbf{H}$ is fibred in groupoids. The objects in \mathbf{hCM} will be called henselian Cohen-Macaulay (hCM) maps.

If $\tilde{h} : \tilde{S} \rightarrow \tilde{T}$ is a CM map and t a k -point in $\mathrm{Spec} \tilde{T}$ with image s in $\mathrm{Spec} \tilde{S}$ then we get a map of the local rings $S = \tilde{S}^h \rightarrow \tilde{T}^h = T$ for the étale topology at t and s given by henselisation. This is a hCM map. Conversely, every hCM map is obtained this way which follows from [21, 18.6.6 and 18.6.10] and [20, 15.4.3 and 12.1.1]. We will call an $\tilde{h} : \tilde{S} \rightarrow \tilde{T}$ in CM which induces $h : S \rightarrow T$ in \mathbf{hCM} a (finite type) representative of h . The dualising module $\omega_{\tilde{h}}$ induces an S -flat finite T -module ω_h called the dualising module for h . Two representatives of h factor through a common étale neighbourhood contained in CM and since the dualising module is functorial for CM the dualising module ω_h is functorial too.

Let \mathbf{mod} denote the category of pairs $(h : S \rightarrow T, \mathcal{N})$ with h in \mathbf{hCM} and \mathcal{N} a finite T -module. Morphisms are defined as for the finite type case and the forgetful functor $\mathbf{mod} \rightarrow \mathbf{hCM}$ makes \mathbf{mod} fibred in abelian categories over \mathbf{hCM} . Let $\mathbf{mod}^{\mathrm{fl}}$ denote the full subcategory of pairs (h, \mathcal{N}) in \mathbf{mod} with \mathcal{N} S -flat and let \mathbf{MCM} denote the full subcategory of $\mathbf{mod}^{\mathrm{fl}}$ where the closed fibre $\mathcal{N} \otimes_S k$ is a maximal Cohen-Macaulay A -module. Let \mathbf{D} denote the full subcategory of \mathbf{MCM} of objects (h, D) with D in $\mathrm{Add}\{\omega_h\}$. All three are FAd subcategories of \mathbf{mod} over \mathbf{hCM} . Any finite T -module \mathcal{N} has finite presentation hence it is induced from a finite module over a representative of T . If \mathcal{N} is contained in one of the subcategories the representative can be assumed to belong to the corresponding finite type subcategory (*loc. cit.*). Similarly all maps in the various fibred categories over \mathbf{H} are induced from maps in the corresponding fibred categories over \mathbf{R} .

Let $\mathbf{L}(\mathbf{C})$ denote the category of (complete) local, noetherian rings with residue field k and local ring homomorphisms. Let \mathbf{ICM} (\mathbf{cCM}) denote the category of local Cohen-Macaulay (\mathbf{ICM}) maps (respectively complete Cohen-Macaulay or \mathbf{cCM} maps) defined analogously as above with (completion of) *essentially of finite type* replacing algebraic. Similar arguing as above makes \mathbf{ICM} (\mathbf{cCM}) fibred in groupoids over $\mathbf{L}(\mathbf{C})$. The definitions of the module categories apply in the local and the complete case too and we use the same notation in all three cases. Again objects and maps are induced from the finite type case.

Either arguing with representatives or applying the proofs for Theorem 5.1 and Corollary 5.2 (with only minor adjustments) we obtain the following.

Corollary 5.6. *Let \mathbf{xCM} denote either \mathbf{hCM} , \mathbf{ICM} or \mathbf{cCM} . The pair $(\mathbf{mod}, \mathbf{MCM})$ of fibred categories over \mathbf{xCM} satisfies BC1-2 and the triple of fibre categories $(\mathbf{mod}_h, \mathbf{MCM}_h, \mathbf{D}_h)$ satisfies AB1-4 for all objects h in \mathbf{xCM} .*

Moreover; the statements (i-v) in Theorem 5.1 and (i-ii) in Corollary 5.2 are valid over \mathbf{xCM} too.

6. MINIMAL APPROXIMATIONS AND SEMI-CONTINUITY OF INVARIANTS

We show that the Cohen-Macaulay approximation and the $\hat{\mathbf{D}}$ -hull in Corollary 5.6 can be chosen to be minimal. Upper semi-continuous invariants on \mathbf{MCM}_A or \mathbf{FID}_A extends to upper semi-continuous invariants on \mathbf{mod}_A . An example is given by the ω_A -ranks in the representing \mathbf{D} -complex.

Lemma 6.1. *Let $S \rightarrow T$ be a homomorphism of noetherian rings and \mathfrak{a} an ideal in S such that $I = \mathfrak{a}T$ is contained in the Jacobson radical of T . Let M and N be finite T -modules. Let $T_n = T/I^{n+1}$, $M_n = T_n \otimes M$ and $N_n = T_n \otimes N$. Suppose there exists a tower of surjections $\{\varphi_n : M_n \rightarrow N_n\}$. Fix any non-negative integer n_0 . Then there exists a T -linear surjection $\psi : M \rightarrow N$ such that $T_{n_0} \otimes \psi = \varphi_{n_0}$. If the φ_n are isomorphisms and N is S -flat then ψ is an isomorphism.*

Proof. Let $\hat{T} = \varprojlim T_n$, $\hat{M} = \varprojlim M_n$ and $\hat{N} = \varprojlim N_n$. We have

$$(6.1.1) \quad \varprojlim \mathrm{Hom}_{T_n}(M_n, N_n) \cong \mathrm{Hom}_{\hat{T}}(\hat{M}, \hat{N}) \cong \hat{T} \otimes \mathrm{Hom}_T(M, N).$$

Hence $\varprojlim \varphi_n = \Sigma r^{(i)} \otimes \beta^{(i)}$ with $r^{(i)} \in \hat{T}$ and $\beta^{(i)} \in \mathrm{Hom}_T(M, N)$. Let $r_{n_0}^{(i)}$ be the image of $r^{(i)}$ under $\hat{T} \rightarrow T_{n_0}$ and choose liftings $t^{(i)}$ in T of $r_{n_0}^{(i)}$. Put $\psi = \Sigma t^{(i)} \beta^{(i)}$. Then $T_{n_0} \otimes \psi = \varphi_{n_0}$. Since $T_{n_0} \otimes \mathrm{coker} \psi = \mathrm{coker} \varphi_{n_0} = 0$, Nakayama's lemma implies $\mathrm{coker} \psi = 0$. Since $T_{n_0} \otimes_T N$ equals $N \otimes_S S/\mathfrak{a}^{n_0+1}$, S -flatness of N implies $T_{n_0} \otimes \ker \psi = \ker \varphi_{n_0}$ and if $\ker \varphi_{n_0} = 0$ then Nakayama's lemma implies $\ker \psi = 0$. \square

Proposition 6.2. *Let \mathbf{xCM} denote either \mathbf{hCM} , \mathbf{ICM} or \mathbf{cCM} , let $h : S \rightarrow T$ be an object in \mathbf{xCM} and let $\xi : \mathcal{L} \rightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{N}$ be an \mathbf{MCM}_h -approximation of \mathcal{N} . Then the following statements are equivalent.*

- (i) *The sequence ξ is a right minimal \mathbf{MCM}_h -approximation.*
- (ii) *There are no surjections $\mathcal{M} \rightarrow \omega_h$ which induces a surjection $\mathcal{L} \rightarrow \omega_h$.*
- (iii) *There are no common ω_h -summand in $\mathcal{L} \rightarrow \mathcal{M}$.*
- (iv) *The closed fibre $\xi \otimes_S k$ is a right minimal \mathbf{MCM}_A -approximation.*
- (v) *The completion $(\xi \otimes_S k)^\wedge$ of the closed fibre is a right minimal $\mathbf{MCM}_{\hat{A}}$ -approximation.*

The analogous statements (i')-(v') for a $\hat{\mathbf{D}}_h^{\mathfrak{a}}$ -hull $\xi' : \mathcal{N} \rightarrow \mathcal{L}' \rightarrow \mathcal{M}'$ are equivalent. In particular:

- (i') *The sequence ξ' is a left minimal $\hat{\mathbf{D}}_h^{\mathfrak{a}}$ -hull.*
- (ii') *There are no surjections $\mathcal{M}' \rightarrow \omega_h$ which induces a surjection $\mathcal{L}' \rightarrow \omega_h$.*

Proof. Suppose there is a surjection $\mathcal{M} \rightarrow \omega_h$ such that the composition $\mathcal{L} \rightarrow \omega_h$ is surjective too. Then the kernels of these maps give a new MCM-approximation of \mathcal{N} by Corollary 5.6. Since a surjection $\mathcal{L} \rightarrow \omega_h$ has to split by Corollary 5.6, ω_h is a common summand in $\mathcal{L} \rightarrow \mathcal{M}$ and π cannot be right minimal.

Let the closed fibre $\xi_{\otimes_S k}$ of the sequence ξ be denoted by $L \rightarrow M \xrightarrow{p} N$. Assume there is a non-surjective endomorphism $\theta : \mathcal{M} \rightarrow \mathcal{M}$ with $\pi\theta = \pi$. Then $\theta_0 = \theta_{\otimes_S k}$ gives a non-surjective endomorphism of M with $p\theta_0 = p$. It follows that the completion $\hat{L} \rightarrow \hat{M} \rightarrow \hat{N}$ is not a right minimal Cohen-Macaulay approximation of \hat{N} . By [23, 1.12.8] there is a common $\omega_{\hat{A}}$ -summand in $\hat{L} \rightarrow \hat{M}$. Let $\varphi : \hat{M} \rightarrow \omega_{\hat{A}}$ denote the projection. By Lemma 6.1 there exists a surjection $\psi : M \rightarrow \omega_A$. The induced map $L \rightarrow \omega_A$ is surjective too. The map ψ lifts to a surjection $\mathcal{M} \rightarrow \omega_h$ (with $\mathcal{L} \rightarrow \omega_h$ surjective) since the canonical map $\text{Hom}_T(\mathcal{M}, \omega_h) \rightarrow \text{Hom}_A(M, \omega_A)$ is surjective by Corollary 2.9. The \hat{D}^{fl} -case is analogous. \square

Corollary 6.3. *Let xCM denote either hCM, ICM or cCM. For any object (h, \mathcal{N}) in the fibred category mod^{fl} over xCM , \mathcal{N} admits a right minimal MCM-approximation and a left minimal \hat{D}^{fl} -hull which remain minimal after base change and which in particular are unique up to non-canonical isomorphism.*

Proof. The existence of a right minimal ξ follows immediately from criterion (iii) in Proposition 6.2. Moreover ξ is right minimal if and only if the closed fibre $\xi_{\otimes_S k}$ is right minimal. Since any base change ξ_1 of ξ has the same closed fibre as ξ , ξ_1 is right minimal if ξ is. \square

Remark 6.4. In [24, 2.3] M. Hashimoto and A. Shida gave essentially the analog of Proposition 6.2 in the absolute case of a Zariski local Cohen-Macaulay ring with a canonical module. They attributed the complete case to Y. Yoshino. Note Miyachi's proof in the cotilting semi-perfect case, see [35, 3.4] (cf. [23, 1.12.8]). In [41, 3.1] A.-M. Simon and J. R. Strooker give a short independent proof. The proof of Proposition 6.2 also works in the Zariski local case in full generality and is different from these (but depends on the complete case).

Since minimal choices of MCM_A -approximations and \hat{D}_A -hulls exist and are unique up to isomorphism any invariant defined for MCM modules or for FID modules is extended to all finite A -modules. Upper semi-continuity of the invariants is also extended as we explain now. First some notation.

Let $h : S \rightarrow T$ be ring homomorphism and \mathcal{N} a finite T -module. If $t \in \text{Spec } T$ has image $s \in \text{Spec } S$, let $\mathcal{N}_{\mathfrak{p}_t}$ denote the localisation of \mathcal{N} at the prime ideal t , let $h_{\mathfrak{p}_t} : S_{\mathfrak{p}_s} \rightarrow T_{\mathfrak{p}_t}$ denote the ring homomorphism obtained by localising, and put $\mathcal{N}(t) = \mathcal{N}_{\mathfrak{p}_t} \otimes_{S_{\mathfrak{p}_s}} k(s)$ which is a $T(t)$ -module; indeed $\mathcal{N}(t) \cong T(t) \otimes_{T_{\mathfrak{p}_t}} \mathcal{N}_{\mathfrak{p}_t}$. If h is a finite type Cohen-Macaulay map, $h_{\mathfrak{p}_t}$ is in ICM.

Suppose μ is an invariant on MCM_A where A is a Cohen-Macaulay local ring with canonical module. Let $_{\text{MCM}}\mu$ denote the induced invariant on mod_A defined by $_{\text{MCM}}\mu(N) = \mu(M)$ where $L \rightarrow M \rightarrow N$ is the minimal Cohen-Macaulay approximation of N . Similarly $_{\text{FID}}\mu(N) = \mu(L)$ for an invariant μ defined on FID_A . Use the minimal hull $N \rightarrow L' \rightarrow M'$ to define $_{\text{FID}}\mu'$ and $_{\text{MCM}}\mu'$.

The following theorem is a major application of what we have done so far.

Theorem 6.5. *Let μ be an additive non-negative numerical invariant defined for maximal Cohen-Macaulay modules or for finite modules of finite injective dimension on a Cohen-Macaulay local ring with canonical module. Assume μ is upper semi-continuous for finite type flat families $(h : S \rightarrow T, \mathcal{M})$ in MCM (or in \hat{D}). Then the induced invariants $_{\text{MCM}}\mu$ and $_{\text{MCM}}\mu'$ (or $_{\text{FID}}\mu$ and $_{\text{FID}}\mu'$) are upper semi-continuous in finite type flat families (h, \mathcal{N}) in mod^{fl} .*

Proof. Given $(h : S \rightarrow T, \mathcal{N})$ in $\mathbf{mod}^{\mathfrak{fl}}$ and $t \in \mathrm{Spec} T$. By Corollary 6.3 there exists open affines $U = \mathrm{Spec} S_1 \subseteq \mathrm{Spec} S$, $V = \mathrm{Spec} T_1 \subseteq \mathrm{Spec} T$ with $t \in V$, $h_1 : S_1 \rightarrow T_1$ induced from h , and a MCM_{h_1} -approximation $\xi : \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N}|_V$ such that the localisation $\xi_{\mathfrak{p}_t}$ is minimal. By Corollary 6.3 $\xi(t)$ is minimal too. Put $n = \mu(\mathcal{M}(t))$. Since μ is upper semi-continuous there is an open $V_n \subseteq V$ containing t such that $\mu(\mathcal{M}(t')) \leq n$ for all $t' \in V_n$. If $L \rightarrow M \rightarrow \mathcal{N}(t')$ is the minimal MCM approximation of $\mathcal{N}(t')$, M is a direct summand of $\mathcal{M}(t')$ by Proposition 6.2, and hence $\mathrm{MCM}\mu(\mathcal{N}(t')) = \mu(M) \leq \mu(\mathcal{M}(t')) \leq n$. \square

Example 6.6. The Betti numbers $\beta_i(M) = \dim \mathrm{Tor}_i^A(M, A/\mathfrak{m}_A)$ are well known upper semi-continuous invariants of finite modules over local rings. By Theorem 6.5 the induced invariants $\mathrm{MCM}\beta_i$, $\mathrm{MCM}\beta'_i$, $\mathrm{FID}\beta_i$ and $\mathrm{FID}\beta'_i$ are upper semi-continuous too.

We now consider some invariants defined in terms of Cohen-Macaulay approximation. If $h : S \rightarrow T$ is in one of the categories of local Cohen-Macaulay maps, a map $\partial : D \rightarrow D'$ of objects in \mathbf{D}_h is said to be *minimal* if $k \otimes_T \partial = 0$. Any module D in \mathbf{D}_h is isomorphic to some $\omega_h^{\oplus n}$ and $\mathrm{End}_T(\omega_h) \cong T$. Hence if ∂ is not minimal then there is a surjection $D' \rightarrow \omega_h$ inducing a surjection $D \rightarrow \omega_h$. By Corollary 5.6 the ω_h splits off from ∂ . Hence any \mathbf{D}_h -complex is homotopy equivalent to one with all differentials being minimal, which is called a minimal \mathbf{D} -complex.

For any module \mathcal{N} in $\mathbf{mod}_h^{\mathfrak{fl}}$ over \mathbf{xCM} we choose a minimal MCM-approximation $\mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N}$ and a minimal $\hat{\mathbf{D}}^{\mathfrak{fl}}$ -hull $\mathcal{N} \rightarrow \mathcal{L}' \rightarrow \mathcal{M}'$ which exist by Corollary 6.3. Spliced with a minimal \mathbf{D} -resolution of \mathcal{L} and a minimal \mathbf{D} -coresolution of \mathcal{M}' we obtain complexes ${}^{-}C^*(\mathcal{N})$, ${}^{+}C^*(\mathcal{N})$ and $D^*(\mathcal{N})$, as defined in Section 2.2, where no differential has any ω_h -summand. We call such choices of these complexes for minimal. They are unique:

Lemma 6.7. *Suppose h is in \mathbf{hCM} , \mathbf{ICM} or \mathbf{cCM} and (h, \mathcal{N}) is in $\mathbf{mod}_h^{\mathfrak{fl}}$. Then minimal choices of ${}^{-}C^*(\mathcal{N})$, ${}^{+}C^*(\mathcal{N})$ and $D^*(\mathcal{N})$ exist and are unique up to non-canonical isomorphisms.*

Proof. Minimal choices ${}^{+}C_1^*$ and ${}^{+}C_2^*$ of coresolutions for \mathcal{N} are by Lemma 2.2 homotopic through chain maps α and β starting with an isomorphism $\mathcal{L}'_1 \cong \mathcal{L}'_2$. If ρ_i are homotopies with $\beta\alpha - \mathrm{id} = \partial\rho_1 + \rho_1\partial$ and $\alpha\beta - \mathrm{id} = \partial\rho_2 + \rho_2\partial$ then tensoring down by $k \otimes_T -$ makes the right hand side of these identities equal to zero by the minimality of the complexes. Hence $\beta\alpha$ and $\alpha\beta$ are surjective endomorphisms, i.e. isomorphisms. The same argument applies to ${}^{-}C^*$ and D^* . \square

For each module \mathcal{N} in $\mathbf{mod}_h^{\mathfrak{fl}}$ we fix a minimal \mathbf{D} -complex $(D^*(\mathcal{N}), \partial^*)$ representing \mathcal{N} . Let $\mathcal{L}' = \mathrm{coker} \partial^{-1}$ and $\mathcal{M} = \ker \partial^0$. Put $\mathrm{Syz}_i^{\omega_h} \mathcal{L}' := \mathrm{coker} \{\partial^{-i-1} : D^{-i-1} \rightarrow D^{-i}\}$ for $i \geq 0$ and $\mathrm{Syz}_i^{\omega_h} \mathcal{M} := \ker \{\partial^i : D^i \rightarrow D^{i+1}\}$ for $i \geq 0$. For any finite T -module \mathcal{N} let the ω_h -rank of \mathcal{N} , denoted $\omega_h\text{-rk}(\mathcal{N})$, be the largest number n with $\omega_h^{\oplus n} \oplus \mathcal{N}' \cong \mathcal{N}$ for some T -module \mathcal{N}' . Since $\mathrm{End}_T(\omega_h) \cong T$ is a local ring, this is a well behaved invariant, cf. [41, Sec. 1.1].

Definition 6.8. Suppose h is in \mathbf{hCM} , \mathbf{ICM} or \mathbf{cCM} and \mathcal{N} is in $\mathbf{mod}_h^{\mathfrak{fl}}$. Define the numbers:

- (i) $d_T^i(\mathcal{N}) := \omega_h\text{-rk}(D^i(\mathcal{N}))$ for all i
- (ii) $\nu_i^T(\mathcal{N}) := \omega_h\text{-rk}(\mathrm{Syz}_i^{\omega_h} \mathcal{L}')$ for $i \geq 0$
- (iii) $\gamma_T(\mathcal{N}) := \omega_h\text{-rk}(\mathcal{M})$

The definition gives well defined invariants of \mathcal{N} by Lemma 6.7. In particular we see that $\nu_0^T(\mathcal{N})$ equals $\omega_h\text{-rk}(\mathcal{L}')$. We also notice that $\omega_h\text{-rk}(\mathrm{Syz}_{-i}^{\omega_h} \mathcal{M}) = 0$ for all $i > 0$ by Proposition 6.2.

The same notation is used for the absolute counterparts of these invariants.

Remark 6.9. In the absolute case with A a Gorenstein local ring, the γ_A -invariant is called Auslander's δ invariant and has been studied in particular by S. Ding. One has that $\gamma_A(A/\mathfrak{m}_A^n) \leq 1$ with equality for $n \gg 0$. The smallest number n with $\gamma_A(A/\mathfrak{m}_A^n) = 1$ is called the *index* of A and Ding has given results and conjectures concerning this invariant, cf. [15]. See also [24].

Let (A, \mathfrak{m}, k) be a noetherian local ring and N a finite A -module. Simon and Strooker introduced the *reduced Bass numbers*

$$(6.9.1) \quad \nu_A^i(N) = \dim_k \operatorname{im}\{\operatorname{Ext}_A^i(k, N) \rightarrow H_{\mathfrak{m}}^i(N)\}$$

and in the case A has a canonical module they showed in [41, 2.6 and 3.10] that

$$(6.9.2) \quad \nu_0^A(N) = \nu_A^{\dim A}(N) \quad \text{and} \quad \nu_i^A(N) = \nu_A^{\dim A-i}(L') \quad \text{for } i \geq 0$$

where L' is the minimal D_A -hull of N . The Canonical Element Conjecture and the Monomial Conjecture are equivalent to $\nu_A^{\dim A}(\mathfrak{b}) \neq 0$ (or equivalently $\gamma_A(A/\mathfrak{b}) = 0$) for certain ideals \mathfrak{b} in any Gorenstein local ring A , see [41, 6.4 and 6.6].

Let \mathbf{P} and $\hat{\mathbf{P}}^{\text{fl}}$ denote the FAdS of finite projective modules, respectively modules of finite projective dimension over \mathbf{xCM} defined as for the finite type case.

Lemma 6.10. *With notation as above:*

- (i) *The functor $F = \operatorname{Hom}_-(\omega_-, -)$ gives exact equivalences of categories fibred in additive categories $\mathbf{D} \simeq \mathbf{P}$ and $\hat{\mathbf{D}}^{\text{fl}} \simeq \hat{\mathbf{P}}^{\text{fl}}$ over \mathbf{xCM} .*
- (ii) *In particular $d^{-i}(\mathcal{N}) = \beta_i(\operatorname{Hom}_T(\omega_h, \mathcal{L}'))$ for all $i \geq 0$.*
- (iii) *$\operatorname{Hom}_T(D^{*\geq 0}(\mathcal{N}), \omega_h)$ gives a minimal free resolution of $\operatorname{Hom}_T(\mathcal{M}, \omega_h)$. In particular $d^i(\mathcal{N}) = \beta_i(\operatorname{Hom}_T(\mathcal{M}, \omega_h))$ for all $i \geq 0$.*

Proof. (i) and (ii) are the local variants of Lemma 5.3. Breaking $\mathcal{M} \hookrightarrow D^{*\geq 0}(\mathcal{N})$ into short exact sequences, (iii) follows from Corollary 5.6. \square

Corollary 6.11. *Let $h : S \rightarrow T$ be a finite type Cohen-Macaulay map and suppose \mathcal{N} is a T -module in $\mathbf{mod}_h^{\text{fl}}$. Then $d^i(\mathcal{N}(t))$ are upper semi-continuous functions in $t \in \operatorname{Spec} T$ for all i .*

Proof. This follows from Theorem 6.5 and Lemma 6.10. \square

Remark 6.12. The invariants ν_0 and γ are not semi-continuous either way, see Example 7.4. Moreover, let \mathcal{L} be in $\hat{\mathbf{D}}_h$ with $L = \mathcal{L} \otimes_S k$. If $\rho_0 : \omega_A \rightarrow L$ is a direct summand, then ρ_0 lifts to a map $\rho : \omega_h \rightarrow \mathcal{L}$, but no lifting ρ of ρ_0 has to split, even if A is a regular ring.

Remark 6.13. One can also define functions of the base. E.g. if $\varphi : \operatorname{Spec} T \rightarrow \operatorname{Spec} S$ denotes the map induced from h and $\mu(\mathcal{N}(t))$ is an upper semi-continuous function of $t \in \operatorname{Spec} T$, then $\varphi_*\mu(\mathcal{N})$ defined by

$$(6.13.1) \quad \varphi_*\mu(\mathcal{N})(s) = \sup_{t \in \varphi^{-1}(s)} \mu(\mathcal{N}(t))$$

is an upper semi-continuous function in $s \in \operatorname{Spec} S$ since φ is an open map.

7. THE FUNDAMENTAL MODULE OF A COHEN-MACAULAY MAP

Example 7.1. Let (A, \mathfrak{m}_A, k) be a Cohen-Macaulay local ring with canonical module ω_A . Let $(G_*, d_*) \rightarrow k$ be a minimal A -free resolution of k and put $\operatorname{Syz}_i = \operatorname{Syz}_i^A k = \operatorname{coker} d_{i+1}$. Suppose $\dim A = d \geq 2$. There are connecting isomorphisms $\operatorname{Ext}_A^1(\operatorname{Syz}_{d-1}, \omega_A) \cong \operatorname{Ext}_A^2(\operatorname{Syz}_{d-2}, \omega_A) \cong \dots \cong \operatorname{Ext}_A^d(k, \omega_A)$ which is isomorphic to k by duality theory. To $1 \in k$ there is hence a non-split short exact sequence

$$(7.1.1) \quad \theta : 0 \rightarrow \omega_A \longrightarrow E_A \xrightarrow{\pi} \operatorname{Syz}_{d-1}^A k \rightarrow 0$$

with E_A uniquely defined up to non-canonical isomorphism. We call E_A for the *fundamental module* of A . We claim that E_A is a maximal Cohen-Macaulay module which implies that (7.1.1) is the minimal MCM approximation of $\text{Syz}_{d-1}^A k$. If we apply $\text{Hom}_A(-, \omega_A)$ to (7.1.1) we obtain the exact sequence

$$(7.1.2) \quad 0 \rightarrow \text{Hom}_A(\text{Syz}_{d-1}^A k, \omega_A) \rightarrow \text{Hom}_A(E_A, \omega_A) \rightarrow \text{End}_A(\omega_A) \xrightarrow{\partial} \text{Ext}_A^1(\text{Syz}_{d-1}^A k, \omega_A) \rightarrow \text{Ext}_A^1(E_A, \omega_A) \rightarrow 0$$

We have $\partial(\text{id}) = \theta$ so ∂ is surjective and $\text{Ext}_A^1(E_A, \omega_A) = 0$. By duality theory (e.g. [11, 3.5.11]) this excludes the possibility $\text{depth } E_A = d - 1$ and we conclude that E_A is a maximal Cohen-Macaulay module. If N is a Cohen-Macaulay module of codimension c we denote $\text{Ext}_A^c(N, \omega_A)$ by N^\vee . Since $\text{End}_A(\omega_A) \cong A$ we get from (7.1.2) a short exact sequence:

$$(7.1.3) \quad 0 \rightarrow \text{Hom}_A(\text{Syz}_{d-1}^A k, \omega_A) \rightarrow E_A^\vee \rightarrow \mathfrak{m}_A \rightarrow 0$$

Since $\text{Ext}_A^i(k, \omega_A) = 0$ for $i \neq d$, $0 \rightarrow G_0^\vee \rightarrow \dots \rightarrow G_{d-2}^\vee \rightarrow \text{Hom}_A(\text{Syz}_{d-1}^A k, \omega_A) \rightarrow 0$ is a finite ω_A -resolution and so (7.1.3) gives the minimal MCM approximation of the maximal ideal. Auslander introduced the fundamental module in the case $d = 2$, see [7].

We can make a relative version of the fundamental module in much the same way. Let $(2)\Delta : \text{CM} \rightarrow \text{CM}$ be the morphism of fibred categories over \mathbf{R} defined by taking the CM map $h : S \rightarrow T$ to the composition $h^{(2)}$ of h with $\iota = 1 \otimes \text{id}_T : T \rightarrow T \otimes_S T$ and taking a morphism $(g, f) : h_1 \rightarrow h_2$ to the composition of two cocartesian squares $(g, f^{\otimes 2})$ as follows:

$$(7.1.4) \quad \begin{array}{ccc} S_1 & \xrightarrow{h_1} & T_1 \\ g \downarrow & & \downarrow f \\ S_2 & \xrightarrow{h_2} & T_2 \end{array} \mapsto \begin{array}{ccccc} S_1 & \xrightarrow{h_1} & T_1 & \xrightarrow{\iota} & T_1 \otimes_{S_1} T_1 \\ g \downarrow & & \downarrow f & & \downarrow f^{\otimes 2} \\ S_2 & \xrightarrow{h_2} & T_2 & \xrightarrow{\iota} & T_2 \otimes_{S_2} T_2 \end{array}$$

There is also a functor $\Delta : \text{CM} \rightarrow \text{CM}$ defined by mapping (g, f) to the rightmost cocartesian square $(f, f^{\otimes 2})$, but it doesn't commute with the forgetful functor $\text{CM} \rightarrow \mathbf{R}$. Let ${}^d\text{CM}$ denote the full subcategory of CM maps of pure relative dimension d . Then ${}^d\text{CM}$ is a fibred subcategory of CM over \mathbf{R} and $(2)\Delta$ and Δ restricts to a morphism $(2)\Delta : {}^d\text{CM} \rightarrow {}^{2d}\text{CM}$ over \mathbf{R} and a functor $\Delta : {}^d\text{CM} \rightarrow {}^d\text{CM}$.

Let $h : S \rightarrow T$ be a finite type CM map of pure relative dimension $d \geq 2$. Consider P in \mathbf{P}_h (see Lemma 5.3) as a $T^{\otimes 2}$ -module by pullback along the multiplication map $\mu : T^{\otimes 2} \rightarrow T$. By Corollary 2.9 $E = \text{Ext}_{T^{\otimes 2}}^d(P, \omega_\iota)$ is flat and finite as T -module, i.e. T -projective. Let P^* denote $\text{Hom}_T(P, T)$. By Corollary 2.9

$$(7.1.5) \quad \text{End}_T(E) \cong \text{Ext}_{T^{\otimes 2}}^d(P, \omega_\iota) \otimes E^* \cong \text{Ext}_{T^{\otimes 2}}^d(P, \omega_\iota \otimes E^*).$$

Combined with the connecting isomorphisms the identity in $\text{End}_T(E)$ corresponds to a canonical extension of $T^{\otimes 2}$ -modules:

$$(7.1.6) \quad 0 \rightarrow \omega_\iota \otimes_T \text{Ext}_{T^{\otimes 2}}^d(P, \omega_\iota)^* \rightarrow E_h(P) \rightarrow \text{Syz}_{d-1}^{T^{\otimes 2}} P \rightarrow 0.$$

Let ${}^d\mathbf{P}$ and ${}^d\text{MCM}$ denote the restriction of \mathbf{P} and MCM to fibred categories over ${}^d\text{CM}$.

Proposition 7.2. *Let $d \geq 2$. The association $(h, P) \mapsto E_h(P)$ in (7.1.6) induces*

- (i) *a functor $E : {}^d\mathbf{P} \rightarrow {}^d\text{MCM}/{}^d\mathbf{P}$ which preserves cocartesian maps and lifts the functor $\Delta : {}^d\text{CM} \rightarrow {}^d\text{CM}$, and*
- (ii) *a morphism $(2)E : {}^d\mathbf{P} \rightarrow {}^{2d}\text{MCM}/{}^{2d}\mathbf{P}$ of fibred categories over \mathbf{R} which lifts $(2)\Delta : {}^d\text{CM} \rightarrow {}^{2d}\text{CM}$.*

Proof. As an extension of T -flat modules, $E_h(P)$ is T -flat. Applying $\text{Hom}_{T^{\otimes 2}}(-, \omega_\iota)$ to (7.1.6) with $E = \text{Ext}_{T^{\otimes 2}}^d(P, \omega_\iota)$ and $\text{Syz}_{d-1} = \text{Syz}_{d-1}^{T^{\otimes 2}}P$ gives an exact sequence

$$(7.2.1) \quad \begin{aligned} 0 \rightarrow \text{Hom}_{T^{\otimes 2}}(\text{Syz}_{d-1}, \omega_\iota) &\rightarrow \text{Hom}_{T^{\otimes 2}}(E_h(P), \omega_\iota) \rightarrow \text{End}_{T^{\otimes 2}}(\omega_\iota) \otimes_T E \xrightarrow{\partial} \\ &\text{Ext}_{T^{\otimes 2}}^1(\text{Syz}_{d-1}, \omega_\iota) \rightarrow \text{Ext}_{T^{\otimes 2}}^1(E_h(P), \omega_\iota) \rightarrow 0 \end{aligned}$$

by Corollary 2.9 and duality theory. In particular there is a canonical isomorphism $\text{Hom}_{T^{\otimes 2}}(\omega_\iota \otimes_T E^*, \omega_\iota) \cong \text{End}_{T^{\otimes 2}}(\omega_\iota) \otimes_T E$. We have that $\text{End}_{T^{\otimes 2}}(\omega_\iota)$ is canonically isomorphic to $T^{\otimes 2}$ and $\partial(t \otimes \xi) = \mu(t) \text{Syz}_{d-1}^{T^{\otimes 2}}(\xi) \in \text{Ext}_{T^{\otimes 2}}^1(\text{Syz}_{d-1}, \omega_\iota)$ where $\text{Syz}_{d-1}^{T^{\otimes 2}}$ is the composition of the connecting isomorphisms. So ∂ is surjective and $\text{Ext}_{T^{\otimes 2}}^1(E_h(P), \omega_\iota) = 0$ by Corollary 2.9. It follows that all fibres of $E_h(P)$ are MCM modules and so $E_h(P)$ is in MCM_ι and (7.1.6) is an MCM_ι -approximation of $\text{Syz}_{d-1}^{T^{\otimes 2}}P$. Let \mathcal{I}_h denote the kernel of $\mu : T^{\otimes 2} \rightarrow T$ and $(-)^{\vee} = \text{Hom}_{T^{\otimes 2}}(-, \omega_\iota)$. From (7.2.1) we get another MCM_ι -approximation

$$(7.2.2) \quad 0 \rightarrow \text{Hom}_{T^{\otimes 2}}(\text{Syz}_{d-1}^{T^{\otimes 2}}P, \omega_\iota) \longrightarrow E_h(P)^{\vee} \longrightarrow \mathcal{I}_h \otimes_T \text{Ext}_{T^{\otimes 2}}^d(P, \omega_\iota) \rightarrow 0.$$

Dualising (7.2.2) induces (7.1.6) since $E_h(P) \cong E_h(P)^{\vee\vee}$ and $\text{Hom}_{T^{\otimes 2}}(\mathcal{I}_h, \omega_\iota) \cong \omega_\iota$. By Theorem 5.1 the image of $E_h(P)^{\vee}$ in $\text{MCM}_\iota/\mathcal{D}_\iota$ is functorial in the $T^{\otimes 2}$ -module $\mathcal{I}_h \otimes E$ which again is contravariantly functorial in P . Since $(-)^{\vee}$ induces an equivalence

$$(7.2.3) \quad \vee : \text{MCM}_\iota/\mathcal{P}_\iota \xleftarrow{\cong} \text{MCM}_\iota^{\text{op}}/\mathcal{D}_\iota^{\text{op}} : \vee$$

we conclude that $E_h(P)$ is functorial in MCM/\mathcal{P} by our functorial choice of extension. This gives (i) and (ii). \square

Corollary 7.3. *For any Cohen-Macaulay map $h : S \rightarrow T$ of pure relative dimension $d \geq 2$ there is a finite $T^{\otimes 2}$ -module $E_h = E_h(T)$ which is faithfully flat along $\iota : T \rightarrow T^{\otimes 2}$ with all fibres being maximal Cohen-Macaulay modules. The association $h \mapsto E_h$ defines a functor ${}^d\text{CM} \rightarrow {}^d\text{MCM}/{}^d\mathcal{P}$ lifting $\Delta : {}^d\text{CM} \rightarrow {}^d\text{CM}$. In particular E_h gives MCM_ι -approximations*

$$0 \rightarrow \omega_\iota \longrightarrow E_h \longrightarrow \text{Syz}_{d-1}^{T^{\otimes 2}}T \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow \text{Hom}_{T^{\otimes 2}}(\text{Syz}_{d-1}^{T^{\otimes 2}}T, \omega_\iota) \longrightarrow E_h^{\vee} \longrightarrow \mathcal{I}_h \rightarrow 0$$

where \mathcal{I}_h is the kernel of the multiplication map $T^{\otimes 2} \rightarrow T$.

Proof. This follows from Proposition 7.2 and (7.2.2) once we have proved the natural isomorphism $\text{Ext}_{T^{\otimes 2}}^d(T, \omega_\iota) \cong T$. Choose a surjection of S -algebras $P \rightarrow T$ with $P = S[t_1, \dots, t_N]$. Recall that ω_ι can be given as $\text{Ext}_{P \otimes T}^{N-d}(T^{\otimes 2}, \omega_{P \otimes T/T})$ where $\omega_{P \otimes T/T} = \wedge^N \Omega_{P \otimes T/T}$. There is a change of rings spectral sequence

$$(7.3.1) \quad E_2^{p,q} = \text{Ext}_{T^{\otimes 2}}^q(T, \text{Ext}_{P \otimes T}^p(T^{\otimes 2}, \omega_{P \otimes T/T})) \Rightarrow \text{Ext}_{P \otimes T}^{p+q}(T, \omega_{P \otimes T/T})$$

which by Corollary 2.9 and duality theory collapses to the canonical isomorphism

$$(7.3.2) \quad \text{Ext}_{T^{\otimes 2}}^d(T, \text{Ext}_{P \otimes T}^{N-d}(T^{\otimes 2}, \omega_{P \otimes T/T})) \cong \text{Ext}_{P \otimes T}^N(T, \omega_{P \otimes T/T}).$$

By [13, 3.5.6] $\text{Ext}_{P \otimes T}^N(T, \omega_{P \otimes T/T})$ is canonically isomorphic to $\omega_{T/T} = T$ as $T^{\otimes 2}$ -module. \square

We will call the module E_h given in Corollary 7.3 for the *fundamental module* of the Cohen-Macaulay map h .

Example 7.4. Let k be an algebraically closed field and A a finite type k -algebra which is Cohen-Macaulay of pure dimension 2. Then the fundamental module $E = E_h$ of $h : k \rightarrow A$ is the maximal Cohen-Macaulay approximation of $I = \ker\{A^{\otimes 2} \rightarrow A\}$ in $\mathbf{mod}_\iota^{\text{fl}}$;

$$(7.4.1) \quad 0 \rightarrow \omega_h \otimes A \longrightarrow E \longrightarrow I \rightarrow 0$$

where $\iota = 1 \otimes \text{id} : A \rightarrow A^{\otimes 2}$ and $\omega_h \cong \omega_A$. Let t in $\text{Spec } A^{\otimes 2}$ be a k -point, and $t_i \in \text{Spec } A$ be the image of t by the i^{th} projection. Let A_i denote A localised at t_i . Let \mathfrak{m}_i be the maximal ideal in A_i . Localising gives a local Cohen-Macaulay map $\iota_{\mathfrak{p}_t} : A_2 \rightarrow (A^{\otimes 2})_{\mathfrak{p}_t}$ and a module $E_{\mathfrak{p}_t}$ in $\mathbf{MCM}_{\iota_{\mathfrak{p}_t}}$. Let $E(t)$ denote base change of $E_{\mathfrak{p}_t}$ to $k(t_2)$. If $t_1 = t_2$ then $I(t) \cong \mathfrak{p}_1$ and $E(t)$ equals the fundamental module E_{A_1} of (7.1.1). If $t_1 = t_2$ is singular, then $\omega\text{-rk}(E(t)) = 0$ while if $t_1 = t_2$ is regular then $E(t) \cong A_1^{\oplus 2}$. If $t_1 \neq t_2$ then $I(t) \cong A_1 \cong E_{A_1}$ and $E(t) \cong A_1^{\oplus 2}$. This shows that $\gamma(I)(t)$ is *not* upper semi-continuous as the d^i -invariants are.

In particular, if A equals $k[x, y, z]/(x^{n+1} - yz)$ with a 2-dimensional rational double point at $\mathfrak{m}_0 = (x, y, z)$, similar considerations give the following table of invariants (note that $\nu_1 = d^{-1}$):

$\iota : A \rightarrow A^{\otimes 2}$	γ	ν_1	d^0	ν_0	$I(t)$
$t_1 = t_2 = 0$ singular point	0	1	4	1	$\mathfrak{m}_0 A_{\mathfrak{m}_0}$
$t_1 = t_2$ non-singular point	2	1	2	0	$\mathfrak{m}_1 A_1$
$t_1 \neq t_2$	1	0	1	1	A_1

8. MAPS OF DEFORMATION FUNCTORS INDUCED BY COHEN-MACAULAY APPROXIMATION

We extend the Cohen-Macaulay approximation over henselian local rings to deformations and obtain maps between the associated deformation functors. We also introduce the appropriate André-Quillen cohomology and links the various cohomologies in a fundamental long-exact sequence.

Fix an object $\xi = (h : S \rightarrow T, \mathcal{N})$ in \mathbf{mod}^{fl} over \mathbf{H} . A *deformation* of ξ is a cocartesian morphism $\alpha_1 : \xi_1 \rightarrow \xi$ in \mathbf{mod}^{fl} . A *map of deformations* $\alpha_1 \rightarrow \alpha_2$ is a cocartesian morphism $\varphi : \xi_1 \rightarrow \xi_2$ in \mathbf{mod}^{fl} such that $\alpha_2 \varphi = \alpha_1$. Deformations and maps of deformations are objects and arrows in the comma category $\mathbf{Def}_\xi := \mathbf{mod}_{\text{coca}}^{\text{fl}}/\xi$ which is fibred in groupoids over the comma category \mathbf{H}/S , see Lemma 3.1 and the proceeding comments. Let the *deformation functor* $\mathbf{Def}_\xi : \mathbf{H}/S \rightarrow \mathbf{Sets}$ be the functor corresponding to the associated groupoid of sets $\overline{\mathbf{Def}}_\xi$. The comma category $\mathbf{Def}_h := \mathbf{hCM}/h$ of deformations of $h : S \rightarrow T$ is also fibred in groupoids over \mathbf{H}/S and we have an obvious factorisation $\mathbf{Def}_\xi \rightarrow \mathbf{hCM}/h \rightarrow \mathbf{H}/S$ which makes \mathbf{Def}_ξ fibred in groupoids over \mathbf{hCM}/h . To ease readability (and by abuse of notation) we put $\mathbf{Def}_{(T, \mathcal{N})}(S_1) = \mathbf{Def}_\xi(S_1 \rightarrow S)$ and $\mathbf{Def}_T(S_1) = \mathbf{Def}_h(S_1 \rightarrow S)$. We also write a *deformation of* (T, \mathcal{N}) meaning a deformation of ξ and likewise in similar situations.

For each object $\xi_i = (h_i, \mathcal{N}_i)$ in \mathbf{mod}^{fl} over \mathbf{H} we choose a minimal MCM-approximation $\pi_i : \mathcal{L}_i \rightarrow \mathcal{M}_i \xrightarrow{\pi_i} \mathcal{N}_i$ and a minimal $\hat{\mathbf{D}}^{\text{fl}}$ -hull $\iota_i : \mathcal{N}_i \xrightarrow{\iota_i} \mathcal{L}'_i \rightarrow \mathcal{M}'_i$ which exist by Corollary 5.6 and Corollary 6.3. For each deformation $\alpha_i : \xi_i \rightarrow \xi_0$ we choose extensions to commutative diagrams of deformations

$$(8.0.2) \quad \begin{array}{ccccc} \mathcal{L}_i & \longrightarrow & \mathcal{M}_i & \xrightarrow{\pi_i} & \mathcal{N}_i \\ \lambda_i \downarrow & & \mu_i \downarrow & & \nu_i \downarrow \\ \mathcal{L}_0 & \longrightarrow & \mathcal{M}_0 & \xrightarrow{\pi_0} & \mathcal{N}_0 \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathcal{N}_i & \xrightarrow{\iota_i} & \mathcal{L}'_i & \longrightarrow & \mathcal{M}'_i \\ \nu_i \downarrow & & \lambda'_i \downarrow & & \mu'_i \downarrow \\ \mathcal{N}_0 & \xrightarrow{\iota_0} & \mathcal{L}'_0 & \longrightarrow & \mathcal{M}'_0 \end{array}$$

as follows: By Corollary 6.3 a base change of π_i by $h_i \rightarrow h_0$ gives a minimal MCM_{h_0} -approximation $\mathcal{M}_i^\# \rightarrow \mathcal{N}_i^\# \xrightarrow{\simeq} \mathcal{N}_0$. By minimality it is isomorphic to π_0 . Choose an isomorphism. Let μ_i be the composition $\mathcal{M}_i \rightarrow \mathcal{M}_i^\# \xrightarrow{\simeq} \mathcal{M}_0$. It is cocartesian. Do similarly for the $\hat{\mathbf{D}}^{\text{fl}}$ -hull. Let these choices be fixed.

The diagrams in (8.0.2) will be called an **MCM-approximation** (denoted π_*) respectively a $\hat{\mathbf{D}}^{\text{fl}}$ -hull (denoted ι_*) of ν_i (this terminology can be justified).

Definition 8.1. There are four maps of deformation functors

$$\sigma_X : \text{Def}_{(h_0, \mathcal{N}_0)} \longrightarrow \text{Def}_{(h_0, X)} : \mathbf{H}/S_0 \longrightarrow \mathbf{Sets}$$

where X can be $\mathcal{M}_0, \mathcal{L}_0, \mathcal{L}'_0$ and \mathcal{M}'_0 given by $[(h_i \rightarrow h_0, \nu_i)] \mapsto [(h_i \rightarrow h_0, x)]$ for x equal to $\mu_i, \lambda_i, \lambda'_i$ and μ'_i in (8.0.2) respectively.

The following lemma implies that these maps are well defined and independent of choices.

Lemma 8.2. *Given two deformations $\nu_{ij} : \mathcal{N}_{ij} \rightarrow \mathcal{N}_{0j}$ ($j = 1, 2$) in \mathbf{mod}^{fl} over $h_{ij} \rightarrow h_{0j}$ in \mathbf{hCM} and MCM-approximations π_{*j} (respectively $\hat{\mathbf{D}}^{\text{fl}}$ -hulls ι_{*j}) of ν_{ij} . Suppose we have a map of short exact sequences $\pi_{01} \rightarrow \pi_{02}$ (respectively $\iota_{01} \rightarrow \iota_{02}$) and a map $\alpha : \mathcal{N}_{i1} \rightarrow \mathcal{N}_{i2}$ lifting $\mathcal{N}_{01} \rightarrow \mathcal{N}_{02}$, i.e. such that the following two diagrams of solid arrows are commutative:*

$$\begin{array}{ccccc} & \mathcal{L}_{i2} & \xrightarrow{\quad} & \mathcal{M}_{i2} & \xrightarrow{\pi_{i2}} & \mathcal{N}_{i2} \\ & \uparrow \lambda_{i2} & \nearrow \gamma & \uparrow \mu_{i2} & \nearrow \alpha & \uparrow \nu_{i2} \\ \mathcal{L}_{i1} & \xrightarrow{\quad} & \mathcal{M}_{i1} & \xrightarrow{\pi_{i1}} & \mathcal{N}_{i1} \\ \downarrow \lambda_{i1} & & \downarrow \mu_{i1} & & \downarrow \nu_{i1} \\ \mathcal{L}_{02} & \xrightarrow{\quad} & \mathcal{M}_{02} & \xrightarrow{\pi_{02}} & \mathcal{N}_{02} \\ \uparrow \beta & & \uparrow \mu_{01} & & \uparrow \pi_{01} \\ \mathcal{L}_{01} & \xrightarrow{\quad} & \mathcal{M}_{01} & \xrightarrow{\pi_{01}} & \mathcal{N}_{01} \end{array} \quad \begin{array}{ccccc} & \mathcal{N}_{i2} & \xrightarrow{\iota_{i2}} & \mathcal{L}'_{i2} & \xrightarrow{\quad} & \mathcal{M}'_{i2} \\ & \uparrow \iota_{i1} & \nearrow \gamma' & \uparrow \lambda'_{i2} & \nearrow & \uparrow \mu'_{i2} \\ \mathcal{N}_{i1} & \xrightarrow{\iota_{i1}} & \mathcal{L}'_{i1} & \xrightarrow{\quad} & \mathcal{M}'_{i1} \\ \downarrow \nu_{i1} & & \downarrow \lambda'_{i1} & & \downarrow \mu'_{i1} \\ \mathcal{N}_{02} & \xrightarrow{\iota_{02}} & \mathcal{L}'_{02} & \xrightarrow{\quad} & \mathcal{M}'_{02} \\ \uparrow \beta' & & \uparrow \mu'_{01} & & \uparrow \iota_{01} \\ \mathcal{N}_{01} & \xrightarrow{\iota_{01}} & \mathcal{L}'_{01} & \xrightarrow{\quad} & \mathcal{M}'_{01} \end{array}$$

Then there exist maps $\gamma : \mathcal{M}_{i1} \rightarrow \mathcal{M}_{i2}$ and $\gamma' : \mathcal{L}'_{i1} \rightarrow \mathcal{L}'_{i2}$ such that the induced left (respectively right) diagram commutes. If α is cocartesian, so are γ and γ' .

Proof. Consider the MCM-approximation case. By applying base changes to the front diagram, we can reduce the problem to the case $h_{i1} \rightarrow h_{01}$ equals $h_{i2} \rightarrow h_{02}$. Then, by Corollary 5.6, there is a lifting $\gamma_1 : \mathcal{M}_{i1} \rightarrow \mathcal{M}_{i2}$ of α . We would like to adjust γ_1 so that it lifts β too. We have that $\mu_{i2}\gamma_1 - \beta\mu_{i1}$ factors through \mathcal{L}_{02} by a map $\tau_i : \mathcal{M}_{i1} \rightarrow \mathcal{L}_{02}$ which induces a unique map $\tau_0 : \mathcal{M}_{01} \rightarrow \mathcal{L}_{02}$ since μ_{i1} is cocartesian. If $\mathcal{D}_* \twoheadrightarrow \mathcal{L}_{i2}$ is a finite D-resolution, then base change gives a finite D-resolution $\mathcal{D}_*^\# \twoheadrightarrow \mathcal{L}_{02}$ and τ_0 lifts to a $\sigma_0 : \mathcal{M}_{01} \rightarrow \mathcal{D}_0^\#$ by Corollary 5.6. By Corollary 2.9 there is a $\sigma : \mathcal{M}_{i1} \rightarrow \mathcal{D}_0$ lifting σ_0 and subtracting the induced map $\mathcal{M}_{i1} \rightarrow \mathcal{M}_{i2}$ from γ_1 gives our desired γ . If α is an isomorphism so is γ by minimality of the approximations π_{ij} . The argument for the $\hat{\mathbf{D}}^{\text{fl}}$ -case is similar. \square

Remark 8.3. There are maps of fibred categories inducing the maps σ_X in Definition 8.1. Two maps $\alpha, \beta : (h_1, \mathcal{N}_1) \rightarrow (h_2, \mathcal{N}_2)$ in $\text{Def}_{(h_0, \mathcal{N}_0)}$ are stably equivalent if $h_1 = h_2$ and $\alpha - \beta$ factors through an object in \mathbf{D} . Let $\underline{\text{Def}}_{(h_0, \mathcal{N}_0)}$ denote the resulting quotient category which is fibred over \mathbf{hCM}/h_0 and over \mathbf{H}/S_0 . Then Lemma 8.2 implies that there are well defined maps of categories fibred in groupoids $\sigma_X : \underline{\text{Def}}_{(h_0, \mathcal{N}_0)} \rightarrow \underline{\text{Def}}_{(h_0, X)}$ for X equal to $\mathcal{M}_0, \mathcal{L}_0, \mathcal{L}'_0$ and \mathcal{M}'_0 . The associated map of functors is σ_X . Note that $\underline{\text{Def}}_{(h_0, \mathcal{N}_0)}$ is different from $\underline{\text{mod}}_{\text{coCa}}^{\text{fl}}/(h_0, \mathcal{N}_0)$. Stably isomorphic modules will in general have different deformation functors. E.g. let $N = A \oplus \omega_A$. If A is not Gorenstein, then one can have $\text{Ext}_A^1(N, N) \neq 0$. But in the stable category N is isomorphic to A which is infinitesimally rigid.

We have the following reformulation. To $(h : S \rightarrow T, \mathcal{N})$ in $\mathbf{mod}^{\mathbf{fl}}$ consider $\Gamma = T \oplus \mathcal{N}$ as a graded S -algebra with T in degree 0 and \mathcal{N} in degree 1. A deformation of graded algebras $\Gamma_1 \rightarrow \Gamma$ over $S_1 \rightarrow S$ in \mathbf{H}/S is equivalent to a deformation (T_1, \mathcal{N}_1) of (T, \mathcal{N}) . More generally, given a homogeneous morphism of \mathbb{Z} -graded rings $S \rightarrow T$ and a graded T -module M , there are *André-Quillen cohomology groups of graded algebras* ${}_0\mathbf{H}^i(S, T, M) = \mathbf{H}^i \mathrm{Hom}_T^{\mathrm{gr}}(L_{T/S}^{\mathrm{gr}}, M)$. Here $L_{T/S}^{\mathrm{gr}}$ is the graded cotangent complex defined as $\Omega_{P/S} \otimes_P T$ where $P = P_S(T)$ is a graded simplicial degree-wise free S -algebra resolution of T and $\Omega_{P/S}$ denotes the Kähler differentials, see [28, IV] for more details (in a more general situation). See also [32]. If $h : S \rightarrow T$ and $p : S_1 \rightarrow S$ are graded ring homomorphisms with p being surjective, then a *lifting* of h ('of T ') along p ('to S_1 ') is a commutative diagram of graded ring homomorphisms

$$(8.3.1) \quad \begin{array}{ccc} T & \xleftarrow{q} & T_1 \\ h \uparrow & & \uparrow h_1 \\ S & \xleftarrow{p} & S_1 \end{array} \quad \text{with } q \otimes S : T_1 \otimes_{S_1} S \cong T \text{ and } \mathrm{Tor}_1^{S_1}(T_1, S) = 0.$$

Two liftings T_1 and T'_1 of T to S_1 are equivalent if there is a graded S_1 -algebra isomorphism $T_1 \cong T'_1$ commuting with q and q' . There is an obstruction theory for liftings of graded algebras in terms of graded André-Quillen cohomology groups.

Proposition 8.4 ([28, 32]). *Given graded ring homomorphisms $S \rightarrow T$ and $p : S_1 \rightarrow S$ with p surjective and $I^2 = 0$ for $I = \ker p$.*

- (i) *There exists an element $\mathrm{ob}(p, T) \in {}_0\mathbf{H}^2(S, T, T \otimes I)$ which is natural in p such that $\mathrm{ob}(p, T) = 0$ if and only if there exists a lifting of T to S_1 .*
- (ii) *If $\mathrm{ob}(p, T) = 0$ then the set of equivalence classes of liftings of T to S_1 is a torsor for ${}_0\mathbf{H}^1(S, T, T \otimes I)$ which is natural in p .*

The element $\mathrm{ob}(p, T)$ is called the *obstruction class* of (p, T) . If the rings and modules are concentrated in degree 0 this equals the ungraded case and the cohomology groups equals the ungraded André-Quillen cohomology $\mathbf{H}^*(S, T, T \otimes I)$.

Proposition 8.5. *Suppose T is an (ungraded) S -algebra and N is a T -module. Let $\Gamma = T \oplus N$ be the graded S -algebra with T in degree 0 and N in degree 1 and let J be a graded Γ -module with graded components $J = J_0 \oplus J_1$ of degree 0 and 1. Then there is a natural long-exact sequence:*

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}_T(N, J_1) &\longrightarrow {}_0\mathrm{Der}_S(\Gamma, J) \longrightarrow \mathrm{Der}_S(T, J_0) \xrightarrow{\partial} \\ \mathrm{Ext}_T^1(N, J_1) &\longrightarrow {}_0\mathbf{H}^1(S, \Gamma, J) \longrightarrow \mathbf{H}^1(S, T, J_0) \xrightarrow{\partial} \mathrm{Ext}_T^2(N, J_1) \rightarrow \dots \end{aligned}$$

Proof. To the graded ring homomorphisms $S \rightarrow T \rightarrow \Gamma$ there is a distinguished triangle of transitivity

$$(8.5.1) \quad L_{\Gamma/T/S}^{\mathrm{gr}} : L_{T/S}^{\mathrm{gr}} \otimes_T \Gamma \longrightarrow L_{\Gamma/S}^{\mathrm{gr}} \longrightarrow L_{\Gamma/T}^{\mathrm{gr}} \longrightarrow L_{T/S}^{\mathrm{gr}} \otimes_T \Gamma[1]$$

in the graded derived category of Γ , see [28, IV 2.3.4]. The (standard) simplicial resolution $P_T(\Gamma)$ equals T in degree 0, the (standard) T -free resolution $F_T(N)$ of the T -module N in degree 1, and higher degree terms, see [28, IV 1.3.2.1]. It follows that $\mathrm{Hom}_T^{\mathrm{gr}}(L_{\Gamma/T}^{\mathrm{gr}}, J) = \mathrm{Hom}_T(F_T(N), J_1)$. Since $L_{T/S}^{\mathrm{gr}} = L_{T/S}$ is concentrated in degree 0, $\mathrm{Hom}_T^{\mathrm{gr}}(L_{T/S}^{\mathrm{gr}} \otimes_T \Gamma, J) = \mathrm{Hom}_T(L_{T/S}, J_0)$. \square

Lemma 8.6. *Let $h : S \rightarrow T$ be a finite type Cohen-Macaulay map and let \mathcal{N} be a T -module in $\mathbf{mod}_h^{\mathbf{fl}}$. Let $\mathcal{L} \rightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{N}$ and $\mathcal{N} \xrightarrow{\iota} \mathcal{L}' \rightarrow \mathcal{M}'$ be an MCM_h -approximation and a $\hat{\mathbf{D}}_h^{\mathbf{fl}}$ -hull of \mathcal{N} . Let X_i denote \mathcal{N} , \mathcal{M} and \mathcal{L}' for $i = 0, 1, 2$*

respectively, and put $\Gamma_i = T \oplus X_i$. Let I be any S -module. Then there are natural maps of short exact sequences of complexes (see (8.5.1))

$$\mathrm{Hom}_{\Gamma_0}^{\mathrm{gr}}(L_{\Gamma_0/T/S}^{\mathrm{gr}}, \Gamma_0 \otimes I) \xrightarrow{\pi^*} \mathrm{Hom}_{\Gamma_1}^{\mathrm{gr}}(L_{\Gamma_1/T/S}^{\mathrm{gr}}, \Gamma_0 \otimes I) \xleftarrow{\pi_*} \mathrm{Hom}_{\Gamma_1}^{\mathrm{gr}}(L_{\Gamma_1/T/S}^{\mathrm{gr}}, \Gamma_1 \otimes I)$$

and

$$\mathrm{Hom}_{\Gamma_0}^{\mathrm{gr}}(L_{\Gamma_0/T/S}^{\mathrm{gr}}, \Gamma_0 \otimes I) \xrightarrow{\iota_*} \mathrm{Hom}_{\Gamma_0}^{\mathrm{gr}}(L_{\Gamma_0/T/S}^{\mathrm{gr}}, \Gamma_2 \otimes I) \xleftarrow{\iota^*} \mathrm{Hom}_{\Gamma_2}^{\mathrm{gr}}(L_{\Gamma_2/T/S}^{\mathrm{gr}}, \Gamma_2 \otimes I).$$

The induced maps of graded André-Quillen cohomology

$$\begin{aligned} {}_0\mathrm{H}^n(\pi_*) : {}_0\mathrm{H}^n(S, \Gamma_1, \Gamma_1 \otimes I) &\longrightarrow {}_0\mathrm{H}^n(S, \Gamma_1, \Gamma_0 \otimes I) \quad \text{and} \\ {}_0\mathrm{H}^n(\iota^*) : {}_0\mathrm{H}^n(S, \Gamma_2, \Gamma_2 \otimes I) &\longrightarrow {}_0\mathrm{H}^n(S, \Gamma_0, \Gamma_2 \otimes I) \end{aligned}$$

are isomorphisms for $n > 0$ and surjections for $n = 0$.

Proof. There is a natural map $L_{\Gamma_1/T/S}^{\mathrm{gr}} \otimes_{\Gamma_1} \Gamma_0 \rightarrow L_{\Gamma_0/T/S}^{\mathrm{gr}}$ of short exact sequences of complexes (cf. [28, II 2.1.1.6]) induced by the graded T -algebra map $\Gamma_1 \rightarrow \Gamma_0$. This gives π^* . Covariance along $\Gamma_1 \otimes I \rightarrow \Gamma_0 \otimes I$ gives π_* . In each (cohomological) degree the rightmost terms are naturally identified with $\mathrm{Hom}_T(L_{T/S}, T \otimes I)$ as in the proof of Proposition 8.5. By Theorem 5.1 and Corollary 2.9 one has $\mathrm{Ext}_T^n(\mathcal{M}, \mathcal{L} \otimes I) = 0$ for $n > 0$ and the ${}_0\mathrm{H}^n(\pi_*)$ -statement follows. The other case is similar. \square

By Lemma 8.6 (and Theorem 5.1) we get induced natural maps for $n > 0$

$$(8.6.1) \quad \sigma_j^n(I) : {}_0\mathrm{H}^n(S, \Gamma_0, \Gamma_0 \otimes_S I) \longrightarrow {}_0\mathrm{H}^n(S, \Gamma_j, \Gamma_j \otimes_S I) \text{ for } j = 1, 2 \quad \text{and}$$

$$(8.6.2) \quad \tau_j^n(I) : \mathrm{Ext}_T^n(X_0, X_0 \otimes_S I) \longrightarrow \mathrm{Ext}_T^n(X_j, X_j \otimes_S I) \text{ for } j = 1, 2.$$

Example 8.7. By elementary diagram chase Lemma 8.6 gives the following:

- (i) If $\pi^* : \mathrm{Ext}_T^n(\mathcal{N}, \mathcal{N} \otimes I) \rightarrow \mathrm{Ext}_T^n(\mathcal{M}, \mathcal{N} \otimes I)$ is an isomorphism for $n = 1$ and injective for $n = 2$ then $\sigma_1^1(I)$ is an isomorphism and $\sigma_1^2(I)$ is injective.
- (ii) If $\iota_* : \mathrm{Ext}_T^n(\mathcal{N}, \mathcal{N} \otimes I) \rightarrow \mathrm{Ext}_T^n(\mathcal{N}, \mathcal{L}' \otimes I)$ is an isomorphism for $n = 1$ and injective for $n = 2$ then $\sigma_2^1(I)$ is an isomorphism and $\sigma_2^2(I)$ is injective.

9. DEFORMING HULLS OF FINITE INJECTIVE DIMENSION

In order to use Artin's Approximation Theorem [5] as extended by D. Popescu [38, 39] we fix an excellent ring \mathcal{O} (see [19, 7.8.2]). We consider the category of local henselian \mathcal{O} -algebras in \mathbf{H} , denoted ${}_{\mathcal{O}}\mathbf{H}$. Fibred categories \mathbf{hCM} and $\mathbf{mod}^{\mathrm{fl}}$ over ${}_{\mathcal{O}}\mathbf{H}$ and \mathbf{Def}_h and \mathbf{Def}_ξ over ${}_{\mathcal{O}}\mathbf{H}/S$ are defined essentially as in Section 8. Our previous constructions and results are valid in this context as well. In particular deformation functors $\mathbf{Def}_{(T, \mathcal{N})} : {}_{\mathcal{O}}\mathbf{H}/S \rightarrow \mathbf{Sets}$ are defined and the MCM-approximation and $\hat{\mathbf{D}}^{\mathrm{fl}}$ -hull induce maps of deformation functors as in Definition 8.1

Definition 9.1. Given a lifting diagram of ungraded ring homomorphisms as in (8.3.1) and a T -module N . Then a *lifting* of N to T_1 is a T_1 -module N_1 with $\mathrm{Tor}_1^{S_1}(N_1, S) = 0$ and a map $N_1 \rightarrow N$ inducing an isomorphism $N_1 \otimes S \cong N$. Two liftings N_1 and N'_1 of N to T_1 are *equivalent* if there is an isomorphism of T_1 -modules $N_1 \cong N'_1$ commuting with the maps to N .

There is an obstruction theory for liftings of modules in terms of Ext-groups.

Proposition 9.2 ([28, IV 3.1.5]). *Given (ungraded) ring homomorphisms as in (8.3.1) with $I^2 = 0$ and a T -module N .*

- (i) *There exists an element $\mathrm{ob}(q, N) \in \mathrm{Ext}_T^2(N, N \otimes I)$ which is natural in q such that $\mathrm{ob}(q, N) = 0$ if and only if there exists a lifting of N to T_1 .*
- (ii) *If $\mathrm{ob}(q, N) = 0$ then the set of equivalence classes of liftings of N to T_1 is a torsor for $\mathrm{Ext}_T^1(N, N \otimes I)$ which is natural in q .*

The element $\text{ob}(q, N)$ is called the *obstruction class* of (q, N) .

Definition 9.3. Let $\mathcal{O}\mathbf{A}/k$ denote the subcategory of artin rings in $\mathcal{O}\mathbf{H}/k$. Let F and G be set-valued functors on $\mathcal{O}\mathbf{H}/k$ (or $\mathcal{O}\mathbf{A}/k$) with $\#F(k) = 1 = \#G(k)$. A map $\varphi : F \rightarrow G$ is *smooth* (formally smooth) if the natural map of sets $f_\varphi : F(S) \rightarrow F(S_0) \times_{G(S_0)} G(S)$ is surjective for all surjections $\pi : S \rightarrow S_0$ in $\mathcal{O}\mathbf{H}/k$ (in $\mathcal{O}\mathbf{A}/k$). An element $\nu \in F(R)$ is *versal* if the induced map $\text{Hom}_{\mathcal{O}\mathbf{H}/k}(R, -) \rightarrow F$ is smooth and R is algebraic as \mathcal{O} -algebra. If the map is bijective then ν is universal. An element $\nu \in F(R)$ (or a formal element, i.e. a tower $\{\nu_n\} \in \varprojlim F(R/\mathfrak{m}_R^{n+1})$) is *formally versal* if the induced map $\text{Hom}_{\mathcal{O}\mathbf{H}/k}(R, -) \rightarrow F$ of functors restricted to $\mathcal{O}\mathbf{A}/k$ is formally smooth. See [3].

Theorem 9.4. Let k be a field and let A be a Cohen-Macaulay local algebraic k -algebra. Let N be a finite A -module. Fix a minimal MCM_A -approximation $L \rightarrow M \rightarrow N$ and a minimal \hat{D}_A -hull $N \rightarrow L' \rightarrow M'$. Consider the map $\sigma_{L'} : \text{Def}_{(A, N)} \rightarrow \text{Def}_{(A, L')}$ of functors $\mathcal{O}\mathbf{H}/k \rightarrow \text{Sets}$ as in Definition 9.1.

- (i) If $\text{grade } N \geq 1$ then $\sigma_{L'}$ is injective.
- (ii) If $\text{grade } N \geq 2$ then $\sigma_{L'}$ restricted to $\mathcal{O}\mathbf{A}/k$ is an isomorphism.
- (iii) If $\text{grade } N \geq 2$ and $\text{Def}_{(A, N)}$ has a versal element then $\sigma_{L'}$ is an isomorphism.

The analogous statements hold for $\sigma_L : \text{Def}_{(A, N)} \rightarrow \text{Def}_{(A, L)}$ if the grade conditions are strengthened by one.

Proof. (i): Given S in $\mathcal{O}\mathbf{H}/k$ and deformations $({}^i h : S \rightarrow {}^i T, {}^i \mathcal{N})$ of (A, N) to S for $i = 1, 2$. Assume that the images $({}^i h, {}^i \mathcal{L}')$ under $\sigma_{L'}$ are isomorphic to $(h : S \rightarrow T, \mathcal{L}')$. Pullback along the isomorphisms of ${}^i h$ with h induce for all these modules deformations over h . We show that the ${}^i \mathcal{N}$ are isomorphic as deformations over h which implies that $\sigma_{L'}$ is injective. Let $S_n = S/\mathfrak{m}_S^{n+1}$, $T_n = T \otimes S_n$ etc.. We construct a tower of isomorphisms $\{\varphi_n : {}^1 \mathcal{N}_n \cong {}^2 \mathcal{N}_n\}$ and conclude by Lemma 6.1 that the deformations are isomorphic. The case $n = 0$ is trivial. Given φ_n and use it to identify the ${}^i \mathcal{N}_n$ and denote them by \mathcal{N}_n . Let $I = \ker(S_{n+1} \rightarrow S_n)$. By Proposition 9.2 there exists an element ξ in $\text{Ext}_{T_n}^1(\mathcal{N}_n, \mathcal{N}_n \otimes I)$ giving the “difference” of the two deformations of \mathcal{N}_n . But $\mathcal{N}_n \otimes I \cong N \otimes I$ and by the edge map isomorphism of (2.7.1) we get $\text{Ext}_{T_n}^i(\mathcal{N}_n, \mathcal{N}_n \otimes I) \cong \text{Ext}_A^i(N, N) \otimes I$ for all i . If $i > 0$ then $\text{Ext}_A^i(L', L') \cong \text{Ext}_A^i(N, L')$ and $\text{Ext}_A^i(N, N) \rightarrow \text{Ext}_A^i(N, L')$ is injective if $\text{grade } N \geq i$ and an isomorphism if $\text{grade } N \geq i + 1$. The obtained injective map $p : \text{Ext}_A^1(N, N) \otimes I \rightarrow \text{Ext}_A^1(L', L') \otimes I$ induces a map of the torsor actions in Proposition 9.2 on the liftings of \mathcal{N}_n and of \mathcal{L}'_n to T_{n+1} . Since the $\hat{D}_{h_{n+1}}^{\mathfrak{A}}$ -hulls of the ${}^i \mathcal{N}_{n+1}$ are isomorphic as deformations, p maps ξ to 0 and so $\xi = 0$ and by Proposition 9.2 the ${}^i \mathcal{N}_{n+1}$ are isomorphic by an isomorphism φ_{n+1} compatible with φ_n .

(ii): By (i) we only need to prove surjectivity. Let $(h : S \rightarrow T, \mathcal{L}')$ be a deformation of (A, L') to S . We proceed by induction on the length of S . Suppose $\sigma_{L'}(T_n, \mathcal{N}_n) = (T_n, \mathcal{L}'_n)$. We find that $\text{ob}(T_{n+1} \rightarrow T_n, \mathcal{N}_n)$ in Proposition 9.2 maps to $\text{ob}(T_{n+1} \rightarrow T_n, \mathcal{L}'_n)$ under $\text{Ext}_{T_n}^2(\mathcal{N}_n, \mathcal{N}_n \otimes I) \rightarrow \text{Ext}_{T_n}^2(\mathcal{L}'_n, \mathcal{L}'_n \otimes I)$ which by the assumption is injective. By assumption there is a lifting $(T_{n+1}, \mathcal{L}'_{n+1})$ to S_{n+1} so $\text{ob}(T_{n+1} \rightarrow T_n, \mathcal{L}'_n) = 0$. Hence there exists a lifting ${}^1 \mathcal{N}_{n+1}$ of \mathcal{N}_n to T_{n+1} . If $\sigma_{L'}({}^1 \mathcal{N}_{n+1}) = {}^1 \mathcal{L}'_{n+1}$ the difference of ${}^1 \mathcal{L}'_{n+1}$ and \mathcal{L}'_{n+1} gives by Proposition 9.2 a $\xi \in \text{Ext}_{T_n}^1(\mathcal{L}'_n, \mathcal{L}'_n \otimes I)$. By assumption $\text{Ext}_{T_n}^1(\mathcal{L}'_n, \mathcal{L}'_n \otimes I)$ is isomorphic to $\text{Ext}_{T_n}^1(\mathcal{N}'_n, \mathcal{N}'_n \otimes I)$. The corresponding element in the latter perturbs ${}^1 \mathcal{N}_{n+1}$ to a lifting \mathcal{N}_{n+1} of \mathcal{N}_n with $\sigma_{L'}(\mathcal{N}_{n+1}) = \mathcal{L}'_{n+1}$.

(iii): Any S in $\mathcal{O}\mathbf{H}/k$ is a direct limit of a filtering system of algebraic \mathcal{O} -algebras in $\mathcal{O}\mathbf{H}/k$. Since $\text{Def}_{(A, L')}$ is locally of finite presentation (A is algebraic and L'

has finite presentation) it is sufficient to prove surjectivity of $\sigma_{L'}$ for S algebraic. Since \mathcal{O} is excellent, so is S by [19, 7.8.3] and [21, 18.7.6]. We proceed as in (ii) and construct a tower of deformations $\{\mathcal{N}_n\}$. If $({}^vT, {}^v\mathcal{N}) \in \text{Def}_{(A,N)}(R)$ is a versal element, there is a corresponding tower of maps $\{f_n : R \rightarrow S_n\}$ such that $({}^vT, {}^v\mathcal{N})$ induces $\{(T_n, \mathcal{N}_n)\}$. We obtain the algebra map $f : R \rightarrow \hat{S}$ which induces a deformation $({}^*T, {}^*\mathcal{N})$ of (A, N) to \hat{S} . We have $\varprojlim {}^*T_n \cong \varprojlim T_n$ and hence also the completion in the maximal ideals gives an isomorphism ${}^*\hat{T} \cong \hat{T}$. By [5, 2.6], [38, 1.3] and [39] there is an isomorphism ${}^*T \cong T \tilde{\otimes}_S \hat{S}$ of deformations of A whereby *T is identified with $T \tilde{\otimes}_S \hat{S}$. The tower of isomorphisms $\{\sigma({}^*\mathcal{N}_n) \cong \mathcal{L}'_n\}$ implies by Lemma 6.1 that there is an isomorphism of deformations $({}^*T, \sigma({}^*\mathcal{N})) \cong ({}^*T, {}^*\mathcal{L}')$ where ${}^*\mathcal{L}' = {}^*T \otimes_T \mathcal{L}'$. To apply Artin's Approximation Theorem we define a functor of S -algebras $F : {}_S\mathbf{H}/k \rightarrow \mathbf{Sets}$ as follows. If \bar{S} is in ${}_S\mathbf{H}/k$ let \bar{T} denote $T \tilde{\otimes}_S \bar{S}$ and let $\bar{\mathcal{L}}'$ denote $\bar{T} \otimes_T \mathcal{L}'$. Then $F(\bar{S})$ is defined as equivalence classes of pairs of maps of \bar{T} -modules $\bar{\xi} = (\bar{\nu} : \bar{\mathcal{N}} \rightarrow N, \bar{\iota} : \bar{\mathcal{N}} \rightarrow \bar{\mathcal{L}}')$ such that $(\bar{S} \rightarrow \bar{T}, \bar{\mathcal{N}})$ is a deformation of (A, N) to \bar{S} . A map $\bar{S} \rightarrow \bar{S}'$ gives a map of pairs by base change. Two pairs, ${}^1\bar{\xi}$ and ${}^2\bar{\xi}$, are equivalent if there is an isomorphism of deformations ${}^1\bar{\mathcal{N}} \cong {}^2\bar{\mathcal{N}}$ commuting with the ${}^j\bar{\iota}$. We show that F is locally of finite presentation. Suppose $\bar{S} = \varinjlim {}^iS$ for a filtered inductive system of algebras in ${}_S\mathbf{H}/k$. Put ${}^iT = T \tilde{\otimes} {}^iS$. Then $\varinjlim {}^iT \cong \bar{T}$ by [19, 7.8.3] and [21, 18.6.14]. Given $\bar{\xi} \in F(\bar{S})$ as above. Since $\bar{\mathcal{N}}$ has finite presentation and since the maps $\bar{\nu}$ and $\bar{\iota}$ can be represented on the finite presentations, there is a finite iT -module ${}^i\mathcal{N}$ and iT -linear maps ${}^i\nu : {}^i\mathcal{N} \rightarrow N$ and ${}^i\iota : {}^i\mathcal{N} \rightarrow {}^i\mathcal{L}' = {}^iT \otimes_T \mathcal{L}'$ inducing $\bar{\xi}$ by base change. We may also assume that ${}^i\mathcal{N}$ is iS -flat. Hence $\varinjlim F({}^iS) \rightarrow F(\bar{S})$ is surjective, and injectivity is similar. Let ${}^*\xi$ denote the element in $F(\hat{S})$ given by ${}^*\mathcal{N} \rightarrow N$ and the \hat{D} -hull ${}^*\mathcal{N} \rightarrow {}^*\mathcal{L}'$. By Artin's Approximation Theorem [5, 1.12], [38, 1.3], [39] there exist a $\xi = (\nu : \mathcal{N} \rightarrow N, \iota : \mathcal{N} \rightarrow \mathcal{L}')$ with $\xi_1 = {}^*\xi_1$. In particular $\iota_0 : \mathcal{N}_0 \rightarrow \mathcal{L}'_0$ is injective and by applying Proposition 2.5 as in Example 2.6 ι is injective and $\text{coker } \iota$ is S -flat. It follows that ι is the \hat{D} -hull of \mathcal{N} and hence $\sigma_{L'}$ is surjective. The L -case is analogous. \square

Consider the groupoid of finite type Cohen-Macaulay maps over noetherian rings $\mathbf{CM} \rightarrow \mathbf{R}$ in Section 5.1. Let \mathcal{O} be any noetherian ring. By abuse of notation let $\mathbf{CM} \rightarrow {}_{\mathcal{O}}\mathbf{R}$ denote the category fibred in groupoids obtained by restriction to the category of noetherian \mathcal{O} -algebras ${}_{\mathcal{O}}\mathbf{R}$. If $\mathcal{O} \rightarrow T^{\circ}$ is a Cohen-Macaulay map then there is a section $s : {}_{\mathcal{O}}\mathbf{R} \rightarrow \mathbf{CM}$ defined by $s : S \mapsto (S \rightarrow T^{\circ} \otimes S)$ and we obtain a fibred subcategory \mathbf{T}° fibred in sets over ${}_{\mathcal{O}}\mathbf{R}$. We restrict \mathbf{mod} , \mathbf{MCM} and \mathbf{D} to \mathbf{T}° and obtain categories fibred in abelian and additive categories over \mathbf{T}° respectively. These restricted fibred categories satisfy the axioms AB1-4 and BC1-2 and we obtain restricted versions of Theorem 5.1 and Corollary 5.2.

For the local version, fix a field k and a Cohen-Macaulay local algebraic k -algebra A . Let $\mathcal{O} \rightarrow T^{\circ}$ be obtained by henselisation of a finite type Cohen-Macaulay map $\tilde{\mathcal{O}} \rightarrow \tilde{T}^{\circ}$ where $\tilde{\mathcal{O}}$ is assumed to be an excellent ring. In particular \mathcal{O} and T° are excellent rings ([19, 7.8.3], [21, 18.7.6]). Assume $T^{\circ}/\mathfrak{m}_{T^{\circ}} \cong k$ and $T^{\circ} \otimes_{\mathcal{O}} k \cong A$. As above there is a section ${}_{\mathcal{O}}\mathbf{H} \rightarrow \mathbf{hCM}$, \mathbf{T}° is the fibred subcategory and we consider deformations in $\mathbf{mod}_{\mathbf{T}^{\circ}}^{\text{fl}}$ of an object $\xi = (S \rightarrow T^{\circ} \tilde{\otimes} S, \mathcal{N})$ and obtain the fibred category of deformations $\text{Def}_{\mathcal{N}}^{T^{\circ}} := (\mathbf{mod}_{\mathbf{T}^{\circ}}^{\text{fl}})_{\text{coca}}/\xi$ over ${}_{\mathcal{O}}\mathbf{H}/S$. The deformation functor $\text{Def}_{\mathcal{N}}^{T^{\circ}} : {}_{\mathcal{O}}\mathbf{H}/S \rightarrow \mathbf{Sets}$ is defined by the associated groupoid of sets $\overline{\text{Def}}_{\mathcal{N}}^{T^{\circ}}$. A special case is given by $\mathcal{O} = k$ and $T^{\circ} = A$.

Corollary 9.5. *With these assumptions in addition to those in Theorem 9.4 consider $\sigma_{L'} : \text{Def}_{\mathcal{N}}^{T^{\circ}} \rightarrow \text{Def}_{L'}^{T^{\circ}}$.*

- (i) *If grade $N \geq 1$ then $\sigma_{L'}$ is injective.*
- (ii) *If grade $N \geq 2$ then $\sigma_{L'}$ restricted to ${}_{\mathcal{O}}A/k$ is an isomorphism.*

(iii) If grade $N \geq 2$ and $\text{Def}_N^{T^o}$ has a versal element then $\sigma_{L'}$ is an isomorphism. The analogous statements hold for $\sigma_L : \text{Def}_N^{T^o} \rightarrow \text{Def}_L^{T^o}$ if the grade conditions are strengthened by one.

Proof. This is not a formal consequence of Theorem 9.4, but the proof is similar. \square

Proposition 9.6. *With general assumptions as in Theorem 9.4:*

- (i) If $Q' = \text{Hom}_A(\omega_A, L')$ and $Q = \text{Hom}_A(\omega_A, L)$ then Q' and Q have finite projective dimension and $\text{Def}_{(A, L')} \cong \text{Def}_{(A, Q')}$ and $\text{Def}_{(A, L)} \cong \text{Def}_{(A, Q)}$.
- (ii) There are natural maps

$$s : \text{Def}_{(A, M)} \longrightarrow \text{Def}_{(A, M')} \quad \text{and} \quad t : \text{Def}_{(A, L')} \longrightarrow \text{Def}_{(A, L)}$$

commuting with the maps $\sigma_X : \text{Def}_{(A, N)} \rightarrow \text{Def}_{(A, X)}$ for X equal to M and M' , and to L' and L , respectively. If A is a Gorenstein ring, then s is an isomorphism.

The analogous statements also hold for the deformation functors $\text{Def}_X^{T^o}$.

Proof. Lemma 6.10 implies (i). For any \mathcal{M} in MCM_h let \mathcal{M}^\vee denote $\text{Hom}_T(\mathcal{M}, \omega_h)$. There is a short exact sequence $M \rightarrow \omega_A^{\oplus n} \rightarrow M'$ giving the short exact sequence $M^\vee \leftarrow A^{\oplus n} \leftarrow (M')^\vee$. The map s is the composition $\text{Def}_{(A, M)} \cong \text{Def}_{(A, M^\vee)} \rightarrow \text{Def}_{(A, (M')^\vee)} \cong \text{Def}_{(A, M')}$, the middle map obtained by taking the syzygy of the deformation. If A is a Gorenstein ring then $\omega_A \cong A$ and there is an inverse $\text{Def}_{(A, M')} \rightarrow \text{Def}_{(A, M)}$ given by the syzygy map.

Fix a minimal short-exact sequence $L \rightarrow \omega_A^{\oplus r} \xrightarrow{\mu} L'$. For a deformation $\lambda' : \mathcal{L}' \rightarrow L'$ there is a lifting of μ to a map $\tilde{\mu} : \omega_h^{\oplus r} \rightarrow \mathcal{L}'$. If \mathcal{L} denotes the kernel of $\tilde{\mu}$ then there is a cocartesian map $\lambda : \mathcal{L} \rightarrow L$ commuting with $\omega_h^{\oplus r} \rightarrow \omega_A^{\oplus r}$. By Lemma 8.2 $\lambda' \mapsto \lambda$ gives a well defined map of deformation functors $t : \text{Def}_{(A, L')} \rightarrow \text{Def}_{(A, L)}$.

There is a commutative diagram of deformations corresponding to (2.0.8), with $\omega_h^{\oplus r} \rightarrow \omega_A^{\oplus r}$ in the D -place, which gives the stated commutativity of maps of deformation functors. \square

Corollary 9.7. *Let A be an Cohen-Macaulay local algebraic k -algebra with residue field k . Suppose $\dim A \geq 2$. Then there exists finite A -modules L' and Q' with $\text{inj.dim } L' = \dim A = \text{pdim } Q'$ and universal deformations $\mathcal{L}' \in \text{Def}_{L'}^A(A)$ and $\mathcal{Q}' \in \text{Def}_{Q'}^A(A)$.*

Proof. Let $h = 1 \otimes \text{id} : A \rightarrow A \tilde{\otimes}_k A = T$ and $\mathcal{N} = A$ be the cyclic T -module defined through the multiplication map. Then $T \otimes_A k \cong A$ and $\mathcal{N} \otimes_A k \cong k$ and this gives a deformation $\mathcal{N} \rightarrow k$ of the residue field of A which is universal. If L' is the minimal \hat{D}_A -hull of the residue field k then $\mathcal{L}' = \sigma_{L'}(\mathcal{N}) \in \text{Def}_{L'}^{T^o}(A)$ is universal by Corollary 9.5. If $Q' = \text{Hom}_A(\omega_A, L')$ then $\text{Hom}_T(\omega_T, \mathcal{L}') \in \text{Def}_{Q'}^A(A)$ is universal by Proposition 9.6. \square

The following result extends A. Ishii's [29, 3.2] somewhat, but the proof is essentially the same.

Proposition 9.8. *Let k be a field and let A be a Gorenstein local algebraic k -algebra. Suppose $L \rightarrow M \rightarrow N$ is the minimal Cohen-Macaulay approximation of a finite A -module N . If $\text{depth } N = \dim A - 1$ then*

$$\sigma_M : \text{Def}_{(A, N)} \longrightarrow \text{Def}_{(A, M)} \quad \text{and} \quad \sigma_M^A : \text{Def}_N^A \longrightarrow \text{Def}_M^A \quad \text{are smooth.}$$

Proof. By assumption $L \cong A^{\oplus r}$. Assume (h_1, \mathcal{M}_1) in $\text{Def}_{(A, M)}(S_1)$ maps to (h, \mathcal{M}) along the surjection $S_1 \rightarrow S$. Assume σ_M maps (h', \mathcal{N}) in $\text{Def}_{(A, N)}(S)$ to (h, \mathcal{M}) . We can assume that $h' = h$ and that the minimal MCM_h -approximation of \mathcal{N} is $\mathcal{L} \xrightarrow{\rho} \mathcal{M} \rightarrow \mathcal{N}$ where $\mathcal{L} \cong T^{\oplus r}$. Let $\mathcal{L}_1 = T_1^{\oplus r}$ and choose a lifting $\rho_1 : \mathcal{L}_1 \rightarrow \mathcal{M}_1$

of ρ . Put $\mathcal{N}_1 := \text{coker } \rho_1$ with its natural map to \mathcal{N} . Then \mathcal{N}_1 is S_1 -flat and $\sigma_M(h_1, \mathcal{N}_1) = (h_1, \mathcal{M}_1)$. \square

Remark 9.9. If $\dim A \geq 1$ and an MCM A -module M has a rank, $r = \text{rk } M$, then there is a short exact sequence $A^r \rightarrow M \rightarrow N$ with N a codimension one Cohen-Macaulay module, cf. [11, 1.4.3]. Hence in the case A is a Gorenstein domain all MCM modules admits MCM_A -approximations by CM modules in codimension one and Proposition 9.8 applies. However, it's not always possible to continue this reduction: If A is a normal Gorenstein complete local ring any MCM A -module M is the MCM_A -approximation of a codimension 2 Cohen-Macaulay module up to stable isomorphism if and only if A is a unique factorisation domain, see [44, 31].

If A is a normal domain there is also a short exact sequence $A^{r-1} \rightarrow M \rightarrow I$ where I is an ideal of grade one or two, see [10, VII 4 Thm. 6]. Again Proposition 9.8 applies in the Gorenstein case. Let U denote the regular locus in $X = \text{Spec } A$. If $T = A \otimes_k S$ for S in ${}_k\mathbf{H}/k$ there is a natural section $A \rightarrow T$. Let U_T denote $U \times_X \text{Spec } T$. Consider the subfunctor $\text{Def}_M^{A, \wedge} \subseteq \text{Def}_M^A$ of deformations \mathcal{M} with trivial induced deformation $\wedge^{\text{rk } M} \mathcal{M}|_{U_T}$, cf. [29]. Assume $\dim A = 2$. Since $\bar{I} := H^0(U, I)$ is a reflexive module, one gets a non-trivial quotient functor $\text{Quot}_{I \subseteq \bar{I}}^{\bar{I}}$ and a map $\text{Quot}_{I \subseteq \bar{I}}^{\bar{I}} \rightarrow \text{Def}_M^{A, \wedge}$. The map is also smooth in the Gorenstein case by the argument in Proposition 9.8 (contained in [29, 3.2]). In particular, if E_A is the fundamental module and $A/\mathfrak{m}_A \cong k$ then $\text{Hom}_{{}_k\mathbf{H}/k}(A, -) \cong \text{Quot}_{\mathfrak{m}_A \subseteq A}^A \cong \text{Def}_{E_A/\mathfrak{m}_A}^A$ gives a mini-versal family for $\text{Def}_{E_A}^{A, \wedge}$ by the MCM approximation in (7.4.1), see [29, 3.4]. Any \mathfrak{m}_A -primary ideal would give a similar result.

Example 9.10. Assume $A/\mathfrak{m}_A \cong k$ and let M denote the minimal MCM approximation of k . It's given as $M \cong \text{Hom}_A(\text{Syz}_d^A(k^\vee), \omega_A)$ where $d = \dim A$, cf. Remark 5.5. One has $k^\vee = \text{Ext}_A^d(k, \omega_A) \cong k$. We apply $\text{Hom}_A(-, \omega_A)$ to the short exact sequence $\text{Syz}^A(\mathfrak{m}_A) \rightarrow A^{\oplus \beta_1} \xrightarrow{(\varepsilon)} \mathfrak{m}_A$. Assume $\dim A = 2$. Since $\text{Ext}_A^1(\mathfrak{m}_A, \omega_A) \cong k$ we obtain the MCM approximation of k :

$$(9.10.1) \quad 0 \rightarrow \omega_A \xrightarrow{(\varepsilon)^t} \omega_A^{\oplus \beta_1} \rightarrow M \rightarrow k \rightarrow 0$$

In particular $\text{rk}(M) = \beta_1 - 1$ and $\mu(M) = t(A) \cdot \beta_1 + 1$ where $t(A)$ is the Cohen-Macaulay type of A . In the case $A = A(m) = k[u^m, u^{m-1}v, \dots, v^m]^h$, the vertex of the cone over the rational normal curve of degree m , which has the indecomposable MCM modules $M_i = (u^i, u^{i-1}v, \dots, v^i)$ for $i = 0, \dots, m-1$, one finds that $M = M_{m-1}^{\oplus m}$. We have

$$(9.10.2) \quad \dim_k \text{Def}_M^A(k[\varepsilon]) = \dim_k \text{Ext}_A^1(M, M) = (m-1) \cdot m^2$$

while $\dim_k \text{Def}_k^A(k[\varepsilon]) = m+1$. Even in the Gorenstein case ($m=2$) the tangent map isn't surjective and so Proposition 9.8 cannot in general be extended to depth $N = \dim A - 2$. For a detailed description of the strata defined by Ishii in [29] of the reduced versal deformation space of M , see [22]. Applying $\text{Hom}_A(k, -)$ to $\mathfrak{m}_A \rightarrow A \rightarrow k$ gives an exact sequence

$$(9.10.3) \quad 0 \rightarrow \text{Ext}_A^1(k, k) \rightarrow \text{Def}_{\mathfrak{m}_A}^A(k[\varepsilon]) \rightarrow k^{t(A)} \rightarrow \text{Ext}_A^2(k, k)$$

since $\text{Ext}_A^1(\mathfrak{m}_A, \mathfrak{m}_A) \cong \text{Ext}_A^2(k, \mathfrak{m}_A)$ and $\dim A = 2$. In the case $A = A(m)$, $N = \mathfrak{m}_A$, the fundamental module E_A gives the MCM approximation and $E_A \cong M_{m-1}^{\oplus 2}$ with $\dim_k \text{Def}_{E_A}^A(k[\varepsilon]) = 4(m-1)$. The conclusion in Proposition 9.8 cannot hold in the non-Gorenstein case $m > 2$.

10. EXISTENCE OF VERSAL ELEMENTS

Let k be a field, A an algebraic k -algebra with $A/\mathfrak{m}_A \cong k$ and N a finite A -module. Without any Cohen-Macaulay condition on A we define a deformation $(h : S \rightarrow T, \mathcal{N})$ of the pair (A, N) to an S in ${}_k\mathbf{H}/k$ as before and obtain the deformation functor $\text{Def}_{(A, N)} : {}_k\mathbf{H}/k \rightarrow \mathbf{Sets}$ as equivalence classes of deformations of pairs.

We say that A is an *isolated singularity over k* if there is a finite type k -algebra A^{ft} with a maximal ideal \mathfrak{m}_0 such that the henselisation $(A^{\text{ft}})_{\mathfrak{m}_0}^{\text{h}}$ is isomorphic to A and which is smooth over k at all points in $\text{Spec } A^{\text{ft}} \setminus \{\mathfrak{m}_0\}$. We say that the pair (A, N) is an *isolated singularity over k* if A is an isolated singularity over k and if $N_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module for all prime ideals $\mathfrak{p} \neq \mathfrak{m}_A$. The following result is a consequence of a result of R. Elkik and an argument of H. von Essen.

Theorem 10.1. *Let (A, N) be an isolated singularity over the field k with A equidimensional. Then $\text{Def}_{(A, N)} : {}_k\mathbf{H}/k \rightarrow \mathbf{Sets}$ has a versal element.*

Proof. We apply [3, 3.2] (cf. [14]) to show the existence of a formally versal element for $\text{Def}_{(A, N)}$. By the finiteness conditions it follows that $\text{Def}_{(A, N)}$ is locally of finite presentation. The condition (S1) holds in general by Proposition 8.4. Let $(h : S \rightarrow T, \mathcal{N})$ be a deformation to S and I a finite S -module. If we can show that $\text{Ext}_T^1(\mathcal{N}, \mathcal{N} \otimes I)$ and $H^1(S, T, T \otimes I)$ are finite S -modules, then condition (S2) holds by Proposition 8.4 and Proposition 8.5.

Let $h^{\text{ft}} : S \rightarrow T^{\text{ft}}$ be a finite type representative of $h : S \rightarrow T$. In particular $(T^{\text{ft}})_{\mathfrak{m}}^{\text{h}} \cong T$ for a maximal ideal \mathfrak{m} with $k(\mathfrak{m}) \cong k$. Put $h_0^{\text{ft}} = h^{\text{ft}} \otimes_S k : k \rightarrow A^{\text{ft}}$. We may assume that h_0^{ft} is smooth in the complement of $\mathfrak{m}_0 = \mathfrak{m}_A A^{\text{ft}}$. The non-smooth locus $V(J)$ of h^{ft} is defined by an ideal $J \subseteq T^{\text{ft}}$ such that the image J_0 of J in A^{ft} is \mathfrak{m}_0 -primary. Put $T_N^{\text{ft}} = T^{\text{ft}}/J^{N+1}$. Since A^{ft}/J_0^{N+1} has finite length, T_N^{ft} is finite over S . By [28, III 3.1.2] $H^i(S, T^{\text{ft}}, T^{\text{ft}} \otimes I)$ has support in $V(J)$ for $i > 0$ and hence is finite over S . André-Quillen homology commutes with direct limits and the cotangent complex is trivial for étale extensions. By [2, III 21] we get $H^i(T^{\text{ft}}, T, T \otimes I) \cong \text{Hom}_T(H_i(T^{\text{ft}}, T, T), T \otimes I) = 0$ for all i . From the transitivity sequence it follows that $H^i(S, T, T \otimes I) \cong H^i(S, T^{\text{ft}}, T^{\text{ft}} \otimes I) \otimes_{T^{\text{ft}}} T$ for all i . Put $T_N = T_N^{\text{ft}} \otimes_{T^{\text{ft}}} T$. By [21, 18.5.10] T_N^{ft} is henselian. It follows that T_N is a quotient of T_N^{ft} and hence $H^1(S, T, T \otimes I)$ is finite as S -module. Similarly (but easier) it follows that $\text{Ext}_T^1(\mathcal{N}, \mathcal{N} \otimes I)$ is finite as S -module.

For effectivity, let N^{ft} be a finite A^{ft} -module representing N such that N^{ft} is locally free at the smooth points. There is a deformation functor $\text{Def}_{(A^{\text{ft}}, N^{\text{ft}})} : {}_k\mathbf{H}/k \rightarrow \mathbf{Sets}$ of base change maps of pairs $(S \rightarrow T^{\text{ft}}, \mathcal{N}^{\text{ft}}) \rightarrow (k \rightarrow A^{\text{ft}}, N^{\text{ft}})$ where T^{ft} is a flat S -algebra of finite type and \mathcal{N}^{ft} is an S -flat finite T^{ft} -module. Base change is given by the standard tensor product. Similarly there is a $\text{Def}_{A^{\text{ft}}} : {}_k\mathbf{H}/k \rightarrow \mathbf{Sets}$. Restricted to ${}_k\mathbf{A}/k$ $\text{Def}_{(A^{\text{ft}}, N^{\text{ft}})}$ satisfies (S1) and (S2). Let $\{(T_n^{\text{ft}}, \mathcal{N}_n^{\text{ft}})\} \in \varprojlim \text{Def}_{(A^{\text{ft}}, N^{\text{ft}})}(S_n)$ be a formally versal formal element (where $S_n = S/\mathfrak{m}_S^{n+1}$). Put $S = \varprojlim S_n$. By [17, Théorème 7, p. 595] (cf. [4, II 5.1]) there exists an element $S \rightarrow T^{\text{ft}}$ in $\text{Def}_{A^{\text{ft}}}(S)$ which induces $\{T_n^{\text{ft}}\}$. Let T' be the henselisation of T^{ft} in the maximal ideal $\mathfrak{m} = (T^{\text{ft}} \rightarrow A)^{-1}(\mathfrak{m}_A)$. Then $S \rightarrow T'$ is a deformation of A . Let T^* be the completion of T' at the ideal $\mathfrak{n} = \mathfrak{m}_S T'$ and let $\mathcal{N}^* = \varprojlim \mathcal{N}_n$. Then \mathcal{N}^* is an S -flat finite T^* -module. Let $J^* \subseteq T^*$ denote the ideal $I(\varphi)$ where φ is a minimal presentation of \mathcal{N}^* . Then J^* defines the locus $V(J^*)$ where \mathcal{N}^* is not locally free. Let $J = \ker(T' \rightarrow T^*/J^*)$. Since T^*/J^* is finite as $S = \hat{S}$ -module, $T'/J \cong T^*/J^*$. The proof of [42, 2.3] works in this situation too (there is a typo in line 5: it should be a direct sum, not a tensor product) and shows that the completion of T' in the ideal $\mathfrak{a} = J \cap \mathfrak{m}_S T'$ equals T^* . Since \mathcal{N}^* is locally free on the complement of

$V(\alpha T^*)$, the conditions in [17, Théorème 3] hold. From this result we obtain a finite T' -module \mathcal{N} inducing \mathcal{N}^* . In particular \mathcal{N} is S -flat.

We claim that the henselisation map $\text{Def}_{(A^{\text{ft}}, N^{\text{ft}})} \rightarrow \text{Def}_{(A, N)}$ is formally smooth. It follows that the element (T', \mathcal{N}) in $\text{Def}_{(A, N)}(S)$ is formally versal. For the claim, put $\Gamma^{\text{ft}} = A^{\text{ft}} \oplus N^{\text{ft}}$ and $\Gamma = A \oplus N$ and let $\pi : S_1 \rightarrow S_0 = S_1/I$ be a small surjection in ${}_k\mathbf{A}/k$. The obstruction $\text{ob}(\pi, \Gamma_0^{\text{ft}}) \in {}_0H^2(k, \Gamma^{\text{ft}}, \Gamma^{\text{ft}}) \otimes_k I$ for lifting a deformation Γ_0^{ft} of Γ^{ft} along π maps to the corresponding obstruction $\text{ob}(\pi, \Gamma_0) \in {}_0H^2(k, \Gamma, \Gamma) \otimes_k I$. The isomorphisms $H^i(S, T, T) \cong H^i(S, T^{\text{ft}}, T^{\text{ft}}) \otimes_{T^{\text{ft}}} T$ for all i implies isomorphisms ${}_0H^i(k, \Gamma^{\text{ft}}, \Gamma^{\text{ft}}) \cong {}_0H^i(k, \Gamma, \Gamma)$ for $i = 1, 2$ as in the beginning of the proof. Smoothness follows by the standard obstruction argument. By [3, 3.2] there is an algebraic k -algebra R and a formally versal element (T, \mathcal{N}) in $\text{Def}_{(A, N)}(R)$.

Finally we apply [3, 3.3] to conclude that the formally versal element (T, \mathcal{N}) is versal. We already have (S1) and (S2). To check [3, 3.3(iii)], let S be algebraic in ${}_k\mathbf{H}/k$, I an ideal in S and put $S^* = \varprojlim S_n$ where $S_n = S/I^{n+1}$. Let ${}^i\xi = ({}^iT, {}^i\mathcal{N})$ for $i = 1, 2$ be two elements in $\text{Def}_{(A, N)}(S^*)$ and $\{\theta_n : {}^1\xi_n \cong {}^2\xi_n\}$ be a tower of isomorphisms between the S_n -truncations. There are finite type representatives ${}^i\xi_n^{\text{ft}} = ({}^iT_n^{\text{ft}}, {}^i\mathcal{N}_n^{\text{ft}})$ of the ${}^i\xi$. By the cohomology argument above one obtains by induction a tower of isomorphisms $\{\theta_n^{\text{ft}} : {}^1\xi_n^{\text{ft}} \cong {}^2\xi_n^{\text{ft}}\}$ inducing $\{\theta_n\}$. Since $\varprojlim \hat{S}/I^{n+1}\hat{S} \cong S^*$ where \hat{S} is the completion of S in the maximal ideal, we can apply [17, Lemme p. 600] to conclude that the henselisations of the ${}^iT_n^{\text{ft}}$ in ${}^iT^{\text{ft}}I$ are isomorphic by an isomorphism lifting $\theta_0 : {}^1T_0 \cong {}^2T_0$. Further henselisation in the maximal ideals gives an isomorphism of deformations ${}^1T \cong {}^2T$. By Lemma 6.1 the isomorphism is extended to an isomorphism of the pairs $\psi : {}^1\xi \cong {}^2\xi$ which lifts θ_0 . By [42, 1.3] condition [3, 3.3(ii)] is unnecessary and we conclude that (T, \mathcal{N}) is versal. \square

Remark 10.2. Let A be an Cohen-Macaulay local algebraic k -algebra and N a finite A -module. We say that N has an *isolated singularity* if $N_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module for all prime ideals $\mathfrak{p} \neq \mathfrak{m}_A$. In that case a similar, but easier argument gives that Def_N^A has a versal element. This is the result [42, 2.4] of von Essen, but for a slightly different fibred category of deformations where henselisation is taken along the closed fibre. However it implies the result in our case, essentially by henselisation at \mathfrak{m}_0 . Corollary 10.3 and 10.4 have obvious analogs for Def_N^A in this case.

Corollary 10.3. *Suppose A is an isolated Cohen-Macaulay singularity over the field k and N is a finite length A -module. Let $L \rightarrow M \rightarrow N$ and $N \rightarrow L' \rightarrow M'$ be the minimal MCM_A -approximation and \hat{D}_A -hull of N respectively. Then:*

- (i) $\text{Def}_{(A, N)}$ has a versal element.
- (ii) If $\dim A \geq 2$ and Q' denotes $\text{Hom}_A(\omega_A, L')$ then

$$\text{Def}_{(A, N)} \cong \text{Def}_{(A, L')} \cong \text{Def}_{(A, Q')}.$$

- (iii) If $\dim A \geq 3$ and Q denotes $\text{Hom}_A(\omega_A, L)$ then

$$\text{Def}_{(A, N)} \cong \text{Def}_{(A, L)} \cong \text{Def}_{(A, Q)}.$$

Proof. This is Theorem 10.1, Theorem 9.4 and Proposition 9.6. \square

Corollary 10.4. *Suppose A is a local algebraic k -algebra which is a Gorenstein normal domain with $\dim A = 2$ and N is a finite torsion-free A -module with an isolated singularity. Let $L \rightarrow M \rightarrow N$ be the minimal MCM_A -approximation of N . Assume k is perfect. Then $\text{Def}_{(A, N)}$ and $\text{Def}_{(A, M)}$ both have versal elements and the map $\text{Def}_{(A, N)} \rightarrow \text{Def}_{(A, M)}$ is smooth.*

Proof. Since k is perfect it follows from [19, 6.7.7 and 6.8.6] that (A, N) and (A, M) are isolated singularities and hence Theorem 10.1 applies. Proposition 9.8 gives smoothness since N torsion-free here is equivalent to N being a first syzygy. \square

Example 10.5. Indeed, if N is the syzygy module of a finite length module then it is torsion-free and locally free on the complement of the singularity.

11. DEFORMING MAXIMAL COHEN-MACAULAY APPROXIMATIONS OF COHEN-MACAULAY MODULES

Let $h : S \rightarrow T$ be a homomorphism of noetherian rings and \mathcal{N} an S -flat finite T -module. If $t \in \text{Spec } T$ with image $s \in \text{Spec } S$ recall that $\mathcal{N}_{\mathfrak{p}_t}$ is the localisation of \mathcal{N} at the prime ideal t and $\mathcal{N}(t) = \mathcal{N}_{\mathfrak{p}_t} \otimes_{S_{\mathfrak{p}_s}} k(s)$. An (h, \mathcal{N}) -sequence (or just an h -sequence if $\mathcal{N} = T$ and an h -regular element if $n = 1$) is a sequence $J = (f_1, \dots, f_n)$ in T such that the image \bar{J} in $T(t)$ is a weak $\mathcal{N}(t)$ -sequence for all closed t and such that $J\mathcal{N} \neq \mathcal{N}$. Applying the Koszul complex $K(J, \mathcal{N})$ as in Example 2.6 one sees that an (h, \mathcal{N}) -sequence is the same as a transversally \mathcal{N} -regular sequence relative to S as defined in [21, 19.2.1]. In particular; J is an (h, \mathcal{N}) -sequence if and only if J is an \mathcal{N} -sequence and $\mathcal{N}/J\mathcal{N}$ is S -flat.

Theorem 11.1. *Let $q : \mathcal{O} \rightarrow T^\circ$ denote the henselisation of a finite type Cohen-Macaulay map with $T^\circ/\mathfrak{m}_{T^\circ} = k$ and $T^\circ \otimes_{\mathcal{O}} k = A$. Suppose $J = (f_1, \dots, f_n)$ is a q -sequence. Put $\bar{T}^\circ = T^\circ/J$, $B = \bar{T}^\circ \otimes_{\mathcal{O}} k$ and let \bar{J} be the image of J in A . Let N be a maximal Cohen-Macaulay B -module and $L \rightarrow M \rightarrow N$ the minimal MCM_A -approximation of N . If $\text{ob}(A/\bar{J}^2 \rightarrow B, N) = 0$, then the composition of maps*

$$\text{Def}_N^{\bar{T}^\circ} \longrightarrow \text{Def}_N^{T^\circ} \longrightarrow \text{Def}_M^{T^\circ}$$

*of functors from ${}_{\mathcal{O}}\mathbf{H}/k$ to **Sets** is injective.*

The existence of a splitting $B \rightarrow A/\bar{J}^2$ implies that $\text{ob}(A/\bar{J}^2, N) = 0$ for all B -modules N since $A/\bar{J}^2 \otimes_B N$ gives a lifting of N to A/\bar{J}^2 .

Let \mathbf{C} be a category. Then $\text{Arr } \mathbf{C}$ denotes the category with objects being arrows in \mathbf{C} and arrows being commutative diagrams of arrows in \mathbf{C} . An endo-functor F on \mathbf{C} induces an endo-functor $\text{Arr } F$ on $\text{Arr } \mathbf{C}$. Let B be a local noetherian ring and \mathbf{P}_B the additive subcategory of projective modules in \mathbf{mod}_B . Let $\underline{\text{Hom}}_B(N, M)$ denote the homomorphisms from N to M in the quotient category $\underline{\text{mod}}_B = \mathbf{mod}_B/\mathbf{P}_B$ i.e. B -homomorphisms modulo the ones factoring through an object in \mathbf{P}_B . For each N in \mathbf{mod}_B we fix a minimal B -free resolution and use it to define the syzygy modules of N . Then the association $N \mapsto \text{Syz}_i^B N$ induces an endo-functor on $\underline{\text{mod}}_B$ for each i , considered by A. Heller [25]. Define $\underline{\text{Ext}}_B^i(N, M)$ as $\underline{\text{Hom}}_B(\text{Syz}_i^B N, M)$ which turns out to be isomorphic to $\text{Ext}_B^i(N, M)$ for all $i > 0$.

Lemma 11.2. *Let A be a local noetherian ring and $I = (f_1, \dots, f_n)$ a regular sequence. Put $B = A/I$ and suppose N, N_1 and N_2 are finite B -modules. Let \bar{M}_i denote $B \otimes \text{Syz}_n^A N_i$.*

- (i) *There is an inclusion $u_N : N \rightarrow \bar{M}_N$ of B -modules with $\bar{M}_N \cong B \otimes \text{Syz}_n^A N$ which induces a functor $u : \underline{\text{mod}}_B \rightarrow \text{Arr } \underline{\text{mod}}_B$.*
- (ii) *The functor u commutes with the B -syzygy functor:*

$$\text{Arr } \text{Syz}_i^B(u_N) = u_{\text{Syz}_i^B N}$$

- (iii) The endo-functor $B \otimes \text{Syz}_n^A(-)$ induces a natural map $\underline{\text{Ext}}_B^i(N_1, N_2) \rightarrow \underline{\text{Ext}}_B^i(\bar{M}_1, \bar{M}_2)$ which makes the following diagram commutative:

$$\begin{array}{ccc} \underline{\text{Ext}}_B^i(N_1, N_2) & \xrightarrow{\quad} & \underline{\text{Ext}}_B^i(\bar{M}_1, \bar{M}_2) \\ & \searrow (u_{N_2})^* \quad \swarrow (u_{N_1})^* & \\ & \underline{\text{Ext}}_B^i(N_1, \bar{M}_2) & \end{array}$$

- (iv) The inclusion $u_N : N \hookrightarrow B \otimes \text{Syz}_n^A N$ splits $\iff \text{ob}(A/J^2 \rightarrow B, N) = 0$.

Remark 11.3. Lemma 11.2 (iv) strengthens [8, 3.6] (in the commutative case).

Proof. (i): Suppose $F_* \rightarrow N$ is a minimal A -free resolution of N . Tensoring the short exact sequence $\text{Syz}_n^A N \xrightarrow{j} F_{n-1} \rightarrow \text{Syz}_{n-1}^A N$ with B gives the exact sequence

$$(11.3.1) \quad 0 \rightarrow \text{Tor}_1^A(B, \text{Syz}_{n-1}^A N) \rightarrow B \otimes \text{Syz}_n^A N \rightarrow \bar{F}_{n-1} \rightarrow B \otimes \text{Syz}_{n-1}^A N \rightarrow 0.$$

We have $\text{Tor}_1^A(B, \text{Syz}_{n-1}^A N) \cong \text{Tor}_n^A(B, N) \cong N$. Let u_N be the inclusion $N \cong \ker(B \otimes_A j) \rightarrow B \otimes \text{Syz}_n^A N$. Then $N \mapsto u_N$ gives a functor of quotient categories.

(ii): Let $p : Q \rightarrow N$ be the minimal B -free cover and $P_* \rightarrow \text{Syz}^B N$ the minimal A -free resolution of the B -syzygy $\ker(p)$. Then there is an A -free resolution $H_* \rightarrow Q$ which is an extension of F_* by P_* . Since $\text{Syz}_n^A B \cong A$, tensoring the short exact sequence of resolutions by B we obtain the commutative diagram with exact rows:

$$(11.3.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Syz}^B N & \longrightarrow & Q & \longrightarrow & N \longrightarrow 0 \\ & & \downarrow u_{\text{Syz}^B N} & & \downarrow & & \downarrow u_N \\ 0 & \longrightarrow & B \otimes \text{Syz}_n^A(\text{Syz}^B N) & \longrightarrow & B^r \oplus Q & \longrightarrow & B \otimes \text{Syz}_n^A N \longrightarrow 0 \end{array}$$

which proves the claim.

(iii): By (ii) it is enough to prove this for $i = 0$. The case $i = 0$ follows from the functoriality in (i).

(iv, \Leftarrow): For the case $n = 1$ see the proof of [8, 3.2]. Assume $n \geq 2$. We follow the proof of [8, 3.6] closely. Let $A_1 = A/(f_1)$. Then $F_*^{(1)} = A_1 \otimes F_{* \geq 1}[1]$ gives a minimal A_1 -free resolution of $A_1 \otimes \text{Syz}^A N$. We have $\text{ob}(A/J^2 \rightarrow B, N) = 0 \Rightarrow \text{ob}(A/(f_1)^2 \rightarrow A_1, N) = 0$ and hence N is a direct summand of $A_1 \otimes \text{Syz}^A N$. Let $G_* \rightarrow N$ be a minimal A_1 -free resolution of N . Then G_* is a direct summand of $F_*^{(1)}$ and hence $\text{Syz}_{n-1}^{A_1} N$ is a direct summand of $\text{Syz}_{n-1}^{A_1}(A_1 \otimes \text{Syz}^A N) = A_1 \otimes \text{Syz}_n^A N$. Tensoring this situation with B gives a commutative diagram:

$$(11.3.3) \quad \begin{array}{ccccccccccc} N & \xrightarrow{u} & B \otimes \text{Syz}_n^A N & \xrightarrow{\bar{j}} & \bar{F}_{n-1} & \longrightarrow & \cdots & \longrightarrow & \bar{F}_1 & \longrightarrow & \bar{F}_0 \longrightarrow N \\ \parallel & & \updownarrow & & \updownarrow & & & & \updownarrow & & \\ N & \xrightarrow{u_1} & B \otimes \text{Syz}_{n-1}^{A_1} N & \xrightarrow{\bar{j}_1} & \bar{G}_{n-2} & \longrightarrow & \cdots & \longrightarrow & \bar{G}_0 & \longrightarrow & N \end{array}$$

Since $\text{ob}(A/J^2 \rightarrow B, N) = 0 \Rightarrow \text{ob}(A_1/(f_2, \dots, f_n)^2 \rightarrow B, N) = 0$ the map u_1 splits by induction on n . So u splits. The other direction follows from [8, 3.6]. \square

Proposition 11.4. Let $h : S \rightarrow T$ be a local Cohen-Macaulay map, $J = (f_1, \dots, f_n)$ an h -sequence, $\bar{h} : S \rightarrow \bar{T} = T/J$ the local Cohen-Macaulay map induced from h , and (\bar{h}, \mathcal{N}) an object in $\text{MCM}_{\bar{h}}$. Let $\xi : \mathcal{L} \rightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{N}$ be the minimal MCM_h -approximation of \mathcal{N} . Then tensoring ξ by \bar{T} gives a 4-term exact sequence

$$0 \rightarrow \mathcal{N} \otimes J/J^2 \longrightarrow \bar{\mathcal{L}} \longrightarrow \bar{\mathcal{M}} \xrightarrow{\bar{\pi}} \mathcal{N} \rightarrow 0$$

which represents the obstruction class $\text{ob}(T/J^2 \rightarrow \bar{T}, \mathcal{N}) \in \text{Ext}_{\bar{T}}^2(\mathcal{N}, \mathcal{N} \otimes J/J^2)$.

Moreover; $\text{ob}(T/J^2 \rightarrow \bar{T}, \mathcal{N}) = 0 \iff \text{ob}(T/J^2 \rightarrow \bar{T}, \mathcal{N}^\vee) = 0 \iff \bar{\pi}$ splits where $\mathcal{N}^\vee = \text{Ext}_T^n(\mathcal{N}, \omega_h)$.

Proof. By Proposition 2.5 $\mathrm{Tor}_i^T(\bar{T}, \mathcal{M}) = \mathrm{H}_i(K(\underline{f}) \otimes \mathcal{M}) = 0$ for $i > 0$. There is a map from the defining short exact sequence $\mathrm{Syz}^T \mathcal{N} \rightarrow F_0 \rightarrow \mathcal{N}$ to ξ lifting $\mathrm{id}_{\mathcal{N}}$. Tensoring with \bar{T} gives a map of 4-term exact sequences with outer terms canonically identified. Hence they represent the same element $\mathrm{ob}(T/J^2 \rightarrow \bar{T}, \mathcal{N})$ in $\mathrm{Ext}_{\bar{T}}^2(\mathcal{N}, \mathcal{N} \otimes J/J^2)$.

By the argument in Remark 5.5 we can assume that ξ is given as $\mathrm{im}(d_n^\vee) \rightarrow (\mathrm{Syz}_n^T \mathcal{N}^\vee)^\vee \rightarrow \mathcal{N}^{\vee\vee}$ where (F_*, d_*) is a minimal T -free resolution of \mathcal{N}^\vee . By Lemma 11.2 $\mathrm{ob}(T/J^2 \rightarrow \bar{T}, \mathcal{N}^\vee) = 0$ if and only if $u : \mathcal{N}^\vee \rightarrow \bar{T} \otimes \mathrm{Syz}_n^T \mathcal{N}^\vee$ splits. But applying $\mathrm{Hom}_{\bar{T}}(-, \omega_{\bar{h}})$ to u gives $\bar{\pi}$ since $\mathcal{N} \cong \mathrm{Ext}_T^n(\mathcal{N}^\vee, \omega_h) \cong \mathrm{Hom}_{\bar{T}}(\mathcal{N}^\vee, \omega_{\bar{h}})$. \square

Remark 11.5. In the absolute Gorenstein case with $n = 1$ this is given in [8, 4.5].

Proof of Theorem 11.1. Given S in ${}_{\mathcal{O}}\mathbf{H}/k$ and let $h : S \rightarrow T$ and $\bar{h} : S \rightarrow \bar{T} = T/JT$ be the induced hCM maps. Let ${}^i\mathcal{N}$ be deformations of N to \bar{h} for $i = 1, 2$ and assume that the minimal MCM_h -approximation modules ${}^i\mathcal{M}$ of ${}^i\mathcal{N}$ are isomorphic as deformations of M . We proceed as in the proof of Theorem 9.4 (i) with $S_n = S/\mathfrak{m}_S^{n+1}$, $\mathcal{N}_n = \mathcal{N} \otimes_S S_n$ etc., construct a tower of isomorphisms $\{\varphi_n : {}^1\mathcal{N} \cong {}^2\mathcal{N}\}$, and conclude by Lemma 6.1 that ${}^1\mathcal{N}$ and ${}^2\mathcal{N}$ are isomorphic as deformations of N . For the induction step we use that the map of torsor actions along $\mathrm{Def}_{\mathcal{N}_n}^{\bar{T}}(S_{n+1}) \rightarrow \mathrm{Def}_{\mathcal{M}_n}^T(S_{n+1})$ is induced by a natural map $p : \mathrm{Ext}_B^1(N, N) \rightarrow \mathrm{Ext}_A^1(M, M)$ which is injective. The map p is given as follows.

Let $\pi : M \rightarrow N$ denote the MCM_A -approximation and $\bar{\pi} : \bar{M} \rightarrow N$ be the B -quotient. Then $\bar{\pi}$ splits by Proposition 11.4. Hence $\bar{\pi}^* : \mathrm{Ext}_B^1(N, N) \hookrightarrow \mathrm{Ext}_B^1(\bar{M}, N)$ splits. Since J is an M -regular sequence, $\mathrm{Ext}_B^1(\bar{M}, N) \cong \mathrm{Ext}_A^1(M, N)$. Since $\mathrm{Ext}_A^i(M, L) = 0$ for all $i > 0$, $\pi_* : \mathrm{Ext}_A^1(M, M) \cong \mathrm{Ext}_A^1(M, N)$. Summarised:

$$(11.5.1) \quad \begin{array}{ccc} \mathrm{Ext}_A^1(M, N) & \xleftarrow{\cong} & \mathrm{Ext}_A^1(M, M) \\ \uparrow \cong & & \\ \mathrm{Ext}_B^1(N, N) & \xhookrightarrow{\bar{\pi}^*} & \mathrm{Ext}_B^1(\bar{M}, N) \end{array}$$

\square

The technique used to prove Theorem 11.1 also gives the following result.

Theorem 11.6. *Let \mathcal{O} and T° be henselian and noetherian local rings and $q : \mathcal{O} \rightarrow T^\circ$ a local and flat ring homomorphism with $T^\circ/\mathfrak{m}_{T^\circ} = k$ and $T^\circ \otimes_{\mathcal{O}} k = A$. Suppose $J = (f_1, \dots, f_n)$ is a q -sequence. Put $\bar{T}^\circ = T^\circ/J$, $B = \bar{T}^\circ \otimes_{\mathcal{O}} k$ and let \bar{J} be the image of J in A . Let N be a finite B -module and let M denote the syzygy module $\mathrm{Syz}_n^A N$.*

If $\mathrm{ob}(A/\bar{J}^2 \rightarrow B, N) = 0$ then the natural map $s : \mathrm{Def}_N^{\bar{T}^\circ} \rightarrow \mathrm{Def}_M^{T^\circ}$ is injective.

Proof. We proceed as in the proof of Theorem 9.4 and 11.1. Given deformations ${}^i\mathcal{N}$ of N to \bar{h} for $i = 1, 2$. They map to ${}^i\mathcal{M} := \mathrm{Syz}_n^T({}^i\mathcal{N})$ which we suppose are isomorphic as deformations of M to h . Then the natural syzygy map $s^1 : \mathrm{Ext}_B^1(N, N) \rightarrow \mathrm{Ext}_A^1(M, M)$ induces the map of torsor actions along s of the infinitesimal extensions. The composition of s^1 with $\mathrm{Ext}_A^1(M, M) \rightarrow \mathrm{Ext}_A^1(M, \bar{M}) \cong \mathrm{Ext}_B^1(\bar{M}, \bar{M})$ commutes with the horizontal map in Lemma 11.2 (iii). But Lemma 11.2 (iv) implies that $(u_N)_*$ is injective, hence s^1 is injective too. Proceeding by induction on \mathfrak{m}_S^{n+1} -truncations of the deformations we construct a tower of isomorphisms and conclude by Lemma 6.1. \square

Remark 11.7. Theorem 11.6 resembles [27, Thm. 1]. However Theorem 11.6 makes a sounder statement in a more general setting and has a more transparent proof. Indeed, the various similar results in [27] can be changed and proved accordingly.

12. THE KODAIRA-SPENCER MAP OF COHEN-MACAULAY APPROXIMATIONS

A modular family of objects is roughly speaking a family where the isomorphism class of the fibre changes non-trivially. The Kodaira-Spencer map makes this idea precise. We consider the Kodaira-Spencer classes and maps for families of pairs (algebra, module) and by invoking the long-exact transitivity sequence we relate them to the corresponding notions for the algebra and the module. Then we show that Cohen-Macaulay approximation of modular families under certain “global” conditions akin to those in Theorem 9.4 and 11.1 produce new modular families.

The following is a graded version of [28, II 2.1.5.7].

Definition 12.1. Let $\mathcal{O} \rightarrow S$ and $S \rightarrow \Gamma$ be graded ring homomorphisms with \mathcal{O} and S concentrated in degree 0. The map $L_{\Gamma/S}^{\text{gr}} \rightarrow L_{S/\mathcal{O}} \otimes_S \Gamma[1]$ in the corresponding distinguished transitivity triangle of (graded) cotangent complexes (see (8.5.1)) is called the *Kodaira-Spencer class* of $\mathcal{O} \rightarrow S \rightarrow \Gamma$.

Composing the Kodaira-Spencer class with the natural augmentation map

$$L_{S/\mathcal{O}} \otimes_S \Gamma[1] \longrightarrow \Omega_{S/\mathcal{O}} \otimes_S \Gamma[1]$$

induces an element $\kappa(\Gamma/S/\mathcal{O}) \in {}_0H^1(S, \Gamma, \Omega_{S/\mathcal{O}} \otimes_S \Gamma)$, the cohomological Kodaira-Spencer class, which is also given as follows. Let $\mathcal{P} = \mathcal{P}_{S/\mathcal{O}}$ denote $S \otimes_{\mathcal{O}} S/I^2$ where I is the kernel of the multiplication map $S \otimes_{\mathcal{O}} S \rightarrow S$. There are two ring homomorphisms j_1 and j_2 from S to \mathcal{P} defined by $j_1 : s \mapsto s \otimes 1$ and $j_2 : s \mapsto 1 \otimes s$. Let $d_{S/\mathcal{O}}$ denote the universal derivation (induced by $j_2 - j_1$). The principal parts of Γ is $\mathcal{P} \otimes_S \Gamma$ (with the j_2 tensor product), which gives an S -algebra extension (via j_1) representing the Kodaira-Spencer class:

$$(12.1.1) \quad \kappa(\Gamma/S/\mathcal{O}) : 0 \rightarrow \Omega_{S/\mathcal{O}} \otimes_S \Gamma \longrightarrow \mathcal{P}_{S/\mathcal{O}} \otimes_S \Gamma \longrightarrow \Gamma \rightarrow 0$$

see [28, III 1.2.6]. Since $\mathcal{P} \otimes_S \Gamma$ has a natural \mathcal{P} -algebra structure, (12.1.1) is also a (graded) algebra lifting of Γ along $\mathcal{P} \rightarrow S$ as in Proposition 8.4. The j_1 -extension $\Gamma \otimes_S \mathcal{P} \rightarrow \Gamma$ is a trivial lifting (split by $\text{id}_{\Gamma} \otimes 1_{\mathcal{P}}$) and the difference in ${}_0H^1(S, \Gamma, \Omega_{S/\mathcal{O}} \otimes_S \Gamma)$ given by Proposition 8.4 (ii) equals $\kappa(\Gamma/S/\mathcal{O})$, see [28, III 2.1.5]. Moreover, the difference $1_{\mathcal{P}} \otimes \text{id}_{\Gamma} - \text{id}_{\Gamma} \otimes 1_{\mathcal{P}}$ induces $d_{S/\mathcal{O}} \otimes 1_{\Gamma}$ (in degree 0) which is mapped to $\kappa(\Gamma/S/\mathcal{O})$ by the connecting homomorphism

$$(12.1.2) \quad \partial : \text{Der}_{\mathcal{O}}(S, \Omega_{S/\mathcal{O}} \otimes_S \Gamma) \longrightarrow {}_0H^1(S, \Gamma, \Omega_{S/\mathcal{O}} \otimes_S \Gamma)$$

in the long-exact transitivity sequence, see [28, III 1.2.6.5 and 1.2.7].

In the special case of $\Gamma = T \oplus \mathcal{N}$, \mathcal{N} a T -module and $S = T$, the transitivity sequence of $\mathcal{O} \rightarrow T \rightarrow \Gamma$ is given in Proposition 8.5. The Kodaira-Spencer class equals $\partial(d_{T/\mathcal{O}}) \in \text{Ext}_T^1(\mathcal{N}, \Omega_{T/\mathcal{O}} \otimes_T \mathcal{N})$ and is called the (cohomological) *Atiyah class* and is denoted by $\text{at}_{T/\mathcal{O}}(\mathcal{N})$, cf. [28, IV 2.3.6-7]. The class is represented by the short exact sequence

$$(12.1.3) \quad \text{at}_{T/\mathcal{O}}(\mathcal{N}) : 0 \rightarrow \Omega_{T/\mathcal{O}} \otimes_T \mathcal{N} \longrightarrow \mathcal{P}_{T/\mathcal{O}} \otimes_T \mathcal{N} \longrightarrow \mathcal{N} \rightarrow 0.$$

The *Kodaira-Spencer map* of $\mathcal{O} \rightarrow S \rightarrow \Gamma$

$$(12.1.4) \quad g^{\Gamma} : \text{Der}_{\mathcal{O}}(S) \longrightarrow {}_0H^1(S, \Gamma, \Gamma)$$

is defined by $D \mapsto f_*^D \kappa(\Gamma/S/\mathcal{O})$ where $f^D : \Omega_{S/\mathcal{O}} \rightarrow S$ corresponds to D . Pushout of (12.1.1) by $f^D \otimes \text{id}_{\Gamma}$ gives the corresponding algebra lifting of Γ along $S[\varepsilon] \rightarrow S$ given by $g^{\Gamma}(D)$.

Proposition 12.2. Let Γ denote the graded S -algebra $T \oplus \mathcal{N}$ where $\mathcal{O} \rightarrow S$ and $S \rightarrow T$ are (ungraded) ring homomorphisms and \mathcal{N} is a T -module. Consider the

transitivity sequence of $S \rightarrow T \xrightarrow{i} \Gamma$ in Proposition 8.5:

$$\dots \rightarrow \operatorname{Der}_S(T, \Omega_{S/\mathcal{O}} \otimes_S T) \xrightarrow{\partial} \operatorname{Ext}_T^1(\mathcal{N}, \Omega_{S/\mathcal{O}} \otimes_S \mathcal{N}) \xrightarrow{u} {}_0\operatorname{H}^1(S, \Gamma, \Omega_{S/\mathcal{O}} \otimes_S \Gamma) \xrightarrow{i^*} \operatorname{H}^1(S, T, \Omega_{S/\mathcal{O}} \otimes_S T) \xrightarrow{\partial} \dots$$

- (i) The map i^* takes the Kodaira-Spencer class $\kappa(\Gamma/S/\mathcal{O})$ to $\kappa(T/S/\mathcal{O})$.
- (ii) Assume $\kappa(T/S/\mathcal{O}) = 0$ and choose an S -algebra splitting $\sigma : T \rightarrow \mathcal{P} \otimes_S T$. Then there is a class $\kappa(\sigma, \mathcal{N}) = \kappa(T/S/\mathcal{O}, \sigma, \mathcal{N}) \in \operatorname{Ext}_T^1(\mathcal{N}, \Omega_{S/\mathcal{O}} \otimes_S \mathcal{N})$ which maps to $\kappa(\Gamma/S/\mathcal{O})$ by u .
- (iii) Let $D(\sigma) \in \operatorname{Der}_{\mathcal{O}}(T, \Omega_{S/\mathcal{O}} \otimes_S T)$ be the derivation corresponding to the splitting σ and for each $D_1 \in \operatorname{Der}_{\mathcal{O}}(S)$ let $X_{\sigma}(D_1)$ denote $f_*^{D_1} D(\sigma) \in \operatorname{Der}_{\mathcal{O}}(T)$. Then

$$f_*^{D_1} \kappa(\sigma, \mathcal{N}) = f_*^{X_{\sigma}(D_1)} \operatorname{at}_{T/\mathcal{O}}(\mathcal{N}) \quad \text{in} \quad \operatorname{Ext}_T^1(\mathcal{N}, \mathcal{N}).$$

Proof. The degree zero part of (12.1.1) gives the image $i^* \kappa(\Gamma/S/\mathcal{O})$ represented by the algebra extension

$$(12.2.1) \quad \kappa(T/S/\mathcal{O}) : \quad 0 \rightarrow \Omega_{S/\mathcal{O}} \otimes_S T \longrightarrow \mathcal{P}_{S/\mathcal{O}} \otimes_S T \longrightarrow T \rightarrow 0.$$

The degree one part is the short exact sequence of $\mathcal{P} \otimes_S T$ -modules

$$(12.2.2) \quad \tilde{\alpha} = \tilde{\alpha}(T/S/\mathcal{O}, \mathcal{N}) : \quad 0 \rightarrow \Omega_{S/\mathcal{O}} \otimes_S \mathcal{N} \longrightarrow \mathcal{P}_{S/\mathcal{O}} \otimes_S \mathcal{N} \longrightarrow \mathcal{N} \rightarrow 0.$$

The splitting σ makes $\tilde{\alpha}$ to a short exact sequence of T -modules which defines $\kappa(T/S/\mathcal{O}, \sigma, \mathcal{N})$.

For (iii) we have $X_{\sigma}(D_1) = f_*^{X_{\sigma}(D_1)} d_{T/\mathcal{O}}$ and the result follows from the commutative diagram

$$(12.2.3) \quad \begin{array}{ccccc} D(\sigma) \in \operatorname{Der}_{\mathcal{O}}(T, \Omega_{S/\mathcal{O}} \otimes_S T) & \xrightarrow{\quad} & \operatorname{Der}_{\mathcal{O}}(T, \Omega_{T/\mathcal{O}}) \ni d_{T/\mathcal{O}} & & \\ \downarrow \partial & \swarrow f_*^{D_1} & \operatorname{Der}_{\mathcal{O}}(T, T) & \nwarrow f_*^{X_{\sigma}(D_1)} & \downarrow \partial \\ & & \downarrow \partial & & \\ & \swarrow f_*^{D_1} & \operatorname{Ext}_T^1(\mathcal{N}, \mathcal{N}) & \nwarrow f_*^{X_{\sigma}(D_1)} & \\ \kappa(\sigma, \mathcal{N}) \in \operatorname{Ext}_T^1(\mathcal{N}, \Omega_{S/\mathcal{O}} \otimes_S \mathcal{N}) & \xrightarrow{\quad} & \operatorname{Ext}_T^1(\mathcal{N}, \Omega_{T/\mathcal{O}} \otimes_T \mathcal{N}) \ni \operatorname{at}_{T/\mathcal{O}}(\mathcal{N}) & & \end{array}$$

where the two outer vertical maps are pointed. \square

We call $\kappa(\sigma, \mathcal{N})$ for the Kodaira-Spencer class of $(T/S/\mathcal{O}, \sigma, \mathcal{N})$. Define the *Kodaira-Spencer map of $(T/S/\mathcal{O}, \sigma, \mathcal{N})$*

$$(12.2.4) \quad g^{(\sigma, \mathcal{N})} : \operatorname{Der}_{\mathcal{O}}(S) \longrightarrow \operatorname{Ext}_T^1(\mathcal{N}, \mathcal{N})$$

by $g^{(\sigma, \mathcal{N})}(D) := (f^D \otimes \operatorname{id})_* \kappa(\sigma, \mathcal{N})$.

In the case $T = S \otimes_{\mathcal{O}} T^o$ we always choose the S -algebra splitting $S \otimes_{\mathcal{O}} T^o \rightarrow \mathcal{P}_{S/\mathcal{O}} \otimes_S S \otimes_{\mathcal{O}} T^o \cong \mathcal{P}_{S/\mathcal{O}} \otimes_{\mathcal{O}} T^o$ given by $s \otimes t \mapsto j_1(s) \otimes t$. In particular $\kappa(T/S/\mathcal{O}) = 0$ and we get a canonical Kodaira-Spencer class $\kappa(\mathcal{N})$ and a corresponding Kodaira-Spencer map $g^{\mathcal{N}}$.

Remark 12.3. There is no reason to believe that $\kappa(\sigma, \mathcal{N})$ maps to $\operatorname{at}_{T/\mathcal{O}}(\mathcal{N})$ in diagram (12.2.3) for any choice of σ . While there is a canonical map of short exact sequences (of S -modules)

$$\begin{array}{ccccccc} \kappa(\sigma, \mathcal{N}) : & 0 & \longrightarrow & \Omega_{S/\mathcal{O}} \otimes_S \mathcal{N} & \longrightarrow & \mathcal{P}_{S/\mathcal{O}} \otimes_S \mathcal{N} & \longrightarrow \mathcal{N} \longrightarrow 0 \\ & & & \downarrow & & \downarrow \tau & \parallel \\ \operatorname{at}_{T/\mathcal{O}}(\mathcal{N}) : & 0 & \longrightarrow & \Omega_{T/\mathcal{O}} \otimes_T \mathcal{N} & \longrightarrow & \mathcal{P}_{T/\mathcal{O}} \otimes_T \mathcal{N} & \longrightarrow \mathcal{N} \longrightarrow 0 \end{array}$$

τ is in general not T -linear. However, in the case $S \rightarrow T$ is smooth then $\Omega_{S/\mathcal{O}} \otimes T \rightarrow \Omega_{T/\mathcal{O}} \rightarrow \Omega_{T/S}$ is split exact and hence there is a non-canonical lifting of the universal derivation in $\text{Der}_{\mathcal{O}}(T, \Omega_{T/\mathcal{O}})$ to $\text{Der}_{\mathcal{O}}(T, \Omega_{S/\mathcal{O}} \otimes T)$ and the corresponding choice of splitting σ makes τ T -linear and so (or by (12.2.3)) $\kappa(\sigma, \mathcal{N})$ maps to $\text{at}_{T/\mathcal{O}}(\mathcal{N})$.

Example 12.4. Another special case is given by the base change of $h : S \rightarrow T$ with itself to $h \otimes T : T \rightarrow T \otimes_S T = T^{\otimes 2}$ and a T -flat $T^{\otimes 2}$ -module \mathcal{N} , cf. Section 7. Then $\kappa(T^{\otimes 2}/T/S, \mathcal{N})$ in $\text{Ext}_{T^{\otimes 2}}^1(\mathcal{N}, \Omega_{T^{\otimes 2}/T} \otimes \mathcal{N})$ equals $\text{at}_{T^{\otimes 2}/T}(\mathcal{N})$. The multiplication map $\mu_{T^{\otimes 2}/T} : \mathcal{P}_{T^{\otimes 2}/T} \rightarrow T^{\otimes 2}$ equals $\text{id}_T \otimes \mu_{T/S} : T \otimes_S \mathcal{P}_{T/S} \rightarrow T^{\otimes 2}$. It follows that $\text{at}_{T^{\otimes 2}/T}(\mathcal{N})$ maps to $\text{at}_{T/S}(\mathcal{N})$ in $\text{Ext}_T^1(\mathcal{N}, \Omega_{T/S} \otimes \mathcal{N})$ by the natural map. If $\mathcal{N} = T$ then $\text{at}_{T/S}(T) = 0$, but in general $\text{at}_{T^{\otimes 2}/T}(T) \neq 0$.

Example 12.5. The transitivity sequence of $\mathcal{O} \rightarrow T \xrightarrow{i} \Gamma$ and J with $\Gamma = T \oplus \mathcal{N}$ and $J = J_0 \oplus J_1$ in Proposition 8.5

$$(12.5.1) \quad 0 \rightarrow \text{Hom}_T(\mathcal{N}, J_1) \rightarrow {}_0\text{Der}_{\mathcal{O}}(\Gamma, J) \xrightarrow{i^*} \text{Der}_{\mathcal{O}}(T, J_0) \rightarrow \text{Ext}_T^1(\mathcal{N}, J_1) \rightarrow \dots$$

suggests the following characterisation. An element $\mathcal{D} \in {}_0\text{Der}_{\mathcal{O}}(\Gamma, J)$ is given by its degree 0 restriction $D := i^*(\mathcal{D}) \in \text{Der}_{\mathcal{O}}(T, J_0)$ and its degree 1 restriction $\nabla_D := \mathcal{D}|_{\mathcal{N}} \in \text{Hom}_{\mathcal{O}}(\mathcal{N}, J_1)$ which should satisfy the following Leibniz rule: For all t in T and n in \mathcal{N}

$$(12.5.2) \quad \nabla_D(tn) = t\nabla_D(n) + D(t)n.$$

With notation as in Proposition 12.2 recall that $\kappa(\Gamma/S/\mathcal{O}) = \partial(d_{S/\mathcal{O}} \otimes 1_T)$ in the transitivity sequence of $\mathcal{O} \rightarrow S \rightarrow \Gamma$:

$$(12.5.3) \quad 0 \rightarrow {}_0\text{Der}_S(\Gamma, \Omega_{S/\mathcal{O}} \otimes \Gamma) \rightarrow {}_0\text{Der}_{\mathcal{O}}(\Gamma, \Omega_{S/\mathcal{O}} \otimes \Gamma) \rightarrow \text{Der}_{\mathcal{O}}(S, \Omega_{S/\mathcal{O}} \otimes \Gamma) \xrightarrow{\partial} {}_0H^1(S, \Gamma, \Omega_{S/\mathcal{O}} \otimes \Gamma) \rightarrow \dots$$

Hence $\kappa(\Gamma/S/\mathcal{O}) = 0$ if and only if there exists a $D \in \text{Der}_{\mathcal{O}}(T, \Omega_{S/\mathcal{O}} \otimes T)$ which restricts to $d_{S/\mathcal{O}} \otimes 1_T$ and a $\nabla_D \in \text{Hom}_{\mathcal{O}}(\mathcal{N}, \Omega_{S/\mathcal{O}} \otimes \mathcal{N})$ satisfying (12.5.2). As a well known special case ($S = T$) we get $\text{at}_{T/\mathcal{O}}(\mathcal{N}) = 0$ if and only if there exists a $\nabla \in \text{Hom}_{\mathcal{O}}(\mathcal{N}, \Omega_{T/\mathcal{O}} \otimes \mathcal{N})$ satisfying (12.5.2) with $D = d_{T/\mathcal{O}} \in \text{Der}_{\mathcal{O}}(T, \Omega_{T/\mathcal{O}})$ (i.e. ∇ is a connection), or equivalently, a graded derivation $\mathcal{D} \in {}_0\text{Der}_{\mathcal{O}}(\Gamma, \Omega_{T/\mathcal{O}} \otimes \Gamma)$ restricting to $d_{T/\mathcal{O}}$. Note that (12.5.1) with $J = \Omega_{T/\mathcal{O}} \otimes \Gamma$ equals (12.5.3) in this case.

Recall the maps of cohomology groups $\sigma_j^1(I)$ and $\tau_j^1(I)$ in (8.6.1) and (8.6.2).

Proposition 12.6. *In addition to the assumptions in Lemma 8.6 suppose $\mathcal{O} \rightarrow S$ is a ring homomorphism. For $j = 1, 2$ the following holds:*

- (i) *The map $\sigma_j^1(\Omega_{S/\mathcal{O}})$ takes $\kappa(\Gamma_0/S/\mathcal{O})$ to $\kappa(\Gamma_j/S/\mathcal{O})$ and the Kodaira-Spencer maps $g^{\Gamma_i} : \text{Der}_{\mathcal{O}}(S) \rightarrow {}_0H^1(S, \Gamma_i, \Gamma_i)$ commute with σ_j^1 , i.e. $\sigma_j^1 g^{\Gamma_0} = g^{\Gamma_j}$.*
- (ii) *Assume $\kappa(T/S/\mathcal{O}) = 0$ and choose an S -algebra splitting $\sigma : T \rightarrow \mathcal{P} \otimes_S T$. Then $\tau_j^1(\Omega_{S/\mathcal{O}})$ maps $\kappa(\sigma, \mathcal{N})$ to $\kappa(\sigma, X_j)$ and the Kodaira-Spencer maps $g^{(\sigma, X_i)} : \text{Der}_{\mathcal{O}}(S) \rightarrow \text{Ext}_T^1(X_i, X_i)$ commute with τ_j^1 , i.e. $\tau_j^1 g^{(\sigma, \mathcal{N})} = g^{(\sigma, X_j)}$.*

Proof. (i): Put $\kappa_j = \kappa(\Gamma_j/S/\mathcal{O})$, $\Omega = \Omega_{S/\mathcal{O}}$ and let $\Gamma(\iota) : \Gamma_0 \rightarrow \Gamma_2$ denote the graded ring homomorphism induced from ι . Then $\Gamma(\iota)$ induces a map of short exact sequences $\kappa_0 \rightarrow \kappa_2$, hence a map of short exact sequences $\Gamma(\iota)_* \kappa_0 \rightarrow \Gamma(\iota)^* \kappa_2$, i.e. $\sigma_2(\Omega)(\kappa_0) = (\Gamma(\iota)^*)^{-1} \Gamma(\iota)_* \kappa_0 = \kappa_2$. The maps $\sigma_2(\Omega)$ and $\sigma_2(S)$ commute with the covariant action of $\text{Der}_{\mathcal{O}}(S)$, hence the second assertion follows from the first. The arguments for the cases $j = 1$ and (ii) are similar. \square

There are corresponding *local* Kodaira-Spencer maps given as follows. Let $t \in \text{Spec } T$ map to $s \in \text{Spec } S$ and consider the localisations $S_{\mathfrak{p}_s} \rightarrow T_{\mathfrak{p}_t}$ and $S_{\mathfrak{p}_s} \rightarrow \Gamma_{\mathfrak{p}_t}$

and the induced map $\mathcal{O} \rightarrow S_{\mathfrak{p}_s}$. The localisation map ${}_0H^1(S, \Gamma, \Omega_{S/\mathcal{O}} \otimes_S \Gamma) \rightarrow {}_0H^1(S_{\mathfrak{p}_s}, \Gamma_{\mathfrak{p}_t}, \Omega_{S_{\mathfrak{p}_s}/\mathcal{O}} \otimes_{S_{\mathfrak{p}_s}} \Gamma_{\mathfrak{p}_t})$ maps $\kappa(\Gamma/S/\mathcal{O})$ to $\kappa(\Gamma_{\mathfrak{p}_t}/S_{\mathfrak{p}_s}/\mathcal{O})$. Let $\bar{\kappa}(\Gamma_{\mathfrak{p}_t}/S_{\mathfrak{p}_s}/\mathcal{O})$ denote the image in ${}_0H^1(S_{\mathfrak{p}_s}, \Gamma_{\mathfrak{p}_t}, \Omega_{S_{\mathfrak{p}_s}/\mathcal{O}} \otimes_{S_{\mathfrak{p}_s}} \Gamma(t))$ by the map induced from $\Gamma_{\mathfrak{p}_t} \rightarrow \Gamma_{\mathfrak{p}_t}/\mathfrak{p}_t \Gamma_{\mathfrak{p}_t} = \Gamma(t)$. Assume that $\Gamma_{\mathfrak{p}_t}$ is $S_{\mathfrak{p}_s}$ -flat. Then the natural base change map

$$(12.6.1) \quad {}_0H^1(k(s), \Gamma(t), \Omega_{S_{\mathfrak{p}_s}/\mathcal{O}} \otimes_{S_{\mathfrak{p}_s}} \Gamma(t)) \longrightarrow {}_0H^1(S_{\mathfrak{p}_s}, \Gamma_{\mathfrak{p}_t}, \Omega_{S_{\mathfrak{p}_s}/\mathcal{O}} \otimes_{S_{\mathfrak{p}_s}} \Gamma(t))$$

is an isomorphism, see [28, II 2.2]. With this identification we define $g^{\Gamma}(t)(D) := (f^D \otimes \text{id}_{\Gamma(t)})_* \bar{\kappa}(\Gamma_{\mathfrak{p}_t}/S_{\mathfrak{p}_s}/\mathcal{O})$ for any D in $\text{Der}_{\mathcal{O}}(S_{\mathfrak{p}_s}, k(s))$ and obtain local Kodaira-Spencer maps at t of Γ , and (similarly) of T , respectively:

$$(12.6.2) \quad g^{\Gamma}(t) : \text{Der}_{\mathcal{O}}(S_{\mathfrak{p}_s}, k(s)) \longrightarrow {}_0H^1(k(s), \Gamma(t), \Gamma(t))$$

$$(12.6.3) \quad g^T(t) : \text{Der}_{\mathcal{O}}(S_{\mathfrak{p}_s}, k(s)) \longrightarrow H^1(k(s), T(t), T(t))$$

commuting with the natural map ${}_0H^1(k(s), \Gamma(t), \Gamma(t)) \rightarrow H^1(k(s), T(t), T(t))$ in Proposition 8.5. Note that if \mathcal{O} is an algebraically closed field then $\text{Der}_{\mathcal{O}}(S_{\mathfrak{p}_s}, k(s))$ is canonically isomorphic to the Zariski tangent space at any closed point $s \in \text{Spec } S$.

Let \mathcal{P} and Ω denote $\mathcal{P}_{S_{\mathfrak{p}_s}/\mathcal{O}}$ and $\Omega_{S_{\mathfrak{p}_s}/\mathcal{O}}$ respectively. Assume $\Gamma = T \oplus \mathcal{N}$. As in the global case we get a graded algebra extension representing $\bar{\kappa}(\Gamma_{\mathfrak{p}_t}/S_{\mathfrak{p}_s}/\mathcal{O})$ which in degree 1 is a short exact sequence $\tilde{\alpha}(t) : \Omega \otimes \mathcal{N}(t) \rightarrow k(s) \otimes \mathcal{P} \otimes \mathcal{N}_{\mathfrak{p}_t} \rightarrow \mathcal{N}(t)$. If $\bar{\kappa}(T_{\mathfrak{p}_t}/S_{\mathfrak{p}_s}/\mathcal{O}) = 0$ we choose an S -algebra splitting $\sigma : T(t) \rightarrow k(s) \otimes \mathcal{P} \otimes \Gamma_{\mathfrak{p}_t}$ and the local Kodaira-Spencer class $\bar{\kappa}(\sigma, \mathcal{N}_{\mathfrak{p}_t})$ in $\text{Ext}_{T(t)}^1(\mathcal{N}(t), \Omega \otimes \mathcal{N}(t))$ is represented by the obtained short exact sequence of $T(t)$ -modules. Then we define the local Kodaira-Spencer map of $(T_{\mathfrak{p}_t}/S_{\mathfrak{p}_s}/\mathcal{O}, \sigma, \mathcal{N}_{\mathfrak{p}_t})$

$$(12.6.4) \quad g^{(\sigma, \mathcal{N})}(t) : \text{Der}_{\mathcal{O}}(S_{\mathfrak{p}_s}, k(s)) \longrightarrow \text{Ext}_{T(t)}^1(\mathcal{N}(t), \mathcal{N}(t))$$

by $g^{(\sigma, \mathcal{N})}(t)(D) := (f^D \otimes \text{id})_* \bar{\kappa}(\sigma, \mathcal{N}_{\mathfrak{p}_t})$.

Similarly, the class $g^{\Gamma}(t)(D)$ is represented by a lifting of graded algebras $\Gamma' \rightarrow \Gamma(t)$ along $k[s][\varepsilon] \rightarrow k(s)$. If the lifting $g^T(t)(D) : T' \rightarrow T(t)$ splits, a choice of splitting makes the short exact sequence $\tilde{\alpha}(t)(D) : \varepsilon \mathcal{N}(t) \rightarrow N' \rightarrow \mathcal{N}(t)$ T -linear and defines $\tilde{\alpha}(t)(D)$ as an extension in the subspace $\text{Ext}_{T(t)}^1(\mathcal{N}(t), \mathcal{N}(t))$ of $\text{Ext}_{T'}^1(\mathcal{N}(t), \mathcal{N}(t))$.

We assume that \mathcal{O} is an algebraically closed field k for the rest of this section.

Definition 12.7. Let $h : S \rightarrow T$ be a local flat map of noetherian k -algebras and \mathcal{N} an S -flat T -module. Put $\Gamma = T \oplus \mathcal{N}$, $A = T \otimes_S k$, $N = \mathcal{N} \otimes_S k$ and $\Gamma(0) = \Gamma \otimes_S k = A \oplus N$. We say that (h, \mathcal{N}) is *locally modular* if the local Kodaira-Spencer map $g^{\Gamma}(0) : \text{Der}_k(S, k) \rightarrow {}_0H^1(k, \Gamma(0), \Gamma(0))$ is injective. If in addition $T = S \tilde{\otimes}_k A$ then \mathcal{N} is locally modular if $g^{\mathcal{N}}(0) : \text{Der}_k(S, k) \rightarrow \text{Ext}_A^1(N, N)$ is injective.

If $h^{\text{ft}} : S \rightarrow T$ is a faithfully flat finite type map of noetherian k -algebras with a k -point $t \in \text{Spec } T$ mapping to $s \in \text{Spec } S$ and \mathcal{N} is an S -flat finite T -module, we say that $(h^{\text{ft}}, \mathcal{N})$ is *modular at t* if the henselisation of $(h^{\text{ft}}, \mathcal{N})$ at t is locally modular. If A is a finite type k -algebra and $T = S \otimes_k A$, then \mathcal{N} is modular at t as T -module if its henselisation at t is locally modular. Let $\nabla(h, \mathcal{N})$ ($\nabla_T(\mathcal{N})$ if $T = S \otimes_k A$) denote the set of k -points $t \in \text{Supp } \mathcal{N}$ where (h, \mathcal{N}) (respectively \mathcal{N} as T -module) is modular.

Corollary 12.8. Let $h : S \rightarrow T$ be a finite type Cohen-Macaulay map of k -algebras and let \mathcal{N} be in mod_h^{fl} . Suppose there is an h -regular element contained in $\text{Ann}_T \mathcal{N}$. Let $\iota : \mathcal{N} \rightarrow \mathcal{L}' \rightarrow \mathcal{M}'$ be a \hat{D}_h^{fl} -hull for \mathcal{N} . Then

$$\nabla(h, \mathcal{N}) = \nabla(h, \mathcal{L}') \quad \text{and} \quad \nabla_T(\mathcal{N}) = \nabla_T(\mathcal{L}') \quad \text{if } T = S \otimes_k A.$$

Proof. Let t be a k -point in $\text{Spec } T$. By Theorem 5.1 $\iota(t)$ is a $\hat{D}_{T(t)}$ -hull for $\mathcal{N}(t)$ and by Proposition 6.2 $\iota(t)$ is minimal if and only if $\iota_{\mathfrak{p}_t}$ is minimal. In particular; the minimal hull of $\mathcal{N}_{\mathfrak{p}_t}$ is a direct summand of $\mathcal{L}'_{\mathfrak{p}_t}$. We therefore assume that $\iota_{\mathfrak{p}_t}$ and hence $\iota(t)$ is minimal.

By the grade condition on $\mathcal{N}(t)$, the map τ_2^1 in (8.6.2) is injective. It implies that the map $\sigma_2^1 : {}_0H^1(k(s), \Gamma_0(t), \Gamma_0(t)) \rightarrow {}_0H^1(k(s), \Gamma_2(t), \Gamma_2(t))$ in (8.6.1) is injective and Proposition 12.6 gives the statement. \square

Example 12.9. Let A be a CM finite type k -algebra and domain of dimension ≥ 2 . Let $h : A \rightarrow T = A^{\otimes 2}$ be the base change by $S = A$ and $\mathcal{N} = A$ be the A -flat T -module defined by the multiplication map $T \rightarrow A$. Let $\Delta \subseteq \text{Spec } T$ denote the closed points on the diagonal and let t be a closed point in $\text{Spec } T$ mapping to s in $\text{Spec } A$. If $t \notin \Delta$ then $\mathcal{N}(t) = 0$. If $t \in \Delta$ then $T(t) \cong A_{\mathfrak{p}_s}$ and $\mathcal{N}(t) \cong k(s)$. The local Kodaira-Spencer class $\bar{\kappa}(\mathcal{N}(t)) \in \text{Ext}_{A_{\mathfrak{p}_s}}^1(k(s), \Omega_{A_{\mathfrak{p}_s}/k} \otimes k(s))$ is represented by

$$(12.9.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{A_{\mathfrak{m}}/k} \otimes k(s) & \longrightarrow & k(s) \otimes \mathcal{P}_{A_{\mathfrak{m}}/k} & \longrightarrow & k(s) \longrightarrow 0 \\ & & \cong \uparrow \delta & & \cong \uparrow \chi & & \parallel \\ 0 & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 & \longrightarrow & A/\mathfrak{m}^2 & \longrightarrow & k(s) \longrightarrow 0 \end{array}$$

(with $\mathfrak{m} = \mathfrak{p}_s$) where $\delta(\bar{x}) = d_{A_{\mathfrak{m}}/k}(x) \otimes 1$ and χ is induced by $1 \otimes j_2$ (note that if $x, y \in \mathfrak{m}$ then $1 \otimes j_2(xy) = 1 \otimes ([j_2(x) - j_1(x)][j_2(y) - j_1(y)]) \in k(s) \otimes I^2$). The local Kodaira-Spencer map $g^{\mathcal{N}}(t)$ is given by the pushout

$$(12.9.2) \quad \varphi \in \text{Hom}_{k(s)}(\mathfrak{m}/\mathfrak{m}^2, k(s)) \longrightarrow \text{Ext}_{A_{\mathfrak{m}}}^1(k(s), k(s)) \ni \varphi_* \bar{\kappa}(\mathcal{N}(t))$$

which is an isomorphism. By Corollary 12.8 we have $\nabla_T(\mathcal{N}) = \nabla_T(\mathcal{L}') = \Delta$. Put $\mathcal{Q}' = \text{Hom}_T(\omega_h, \mathcal{L}')$. By Proposition 9.6 also the local Kodaira-Spencer map $g^{\mathcal{Q}'}(t)$ is injective for $t \in \Delta$. Hence $\nabla_T(\mathcal{Q}') = \Delta$. Note that $\mathcal{L}'(t)$ and $\mathcal{Q}'(t)$ are rigid for $t \notin \Delta$.

Corollary 12.10. Suppose $T = S \otimes_k A$ for a finite type Cohen-Macaulay k -algebra A . Let $J = (f_1, \dots, f_n)$ be an A -sequence, put $B = A/J$ and let $\bar{h} : S \rightarrow \bar{T} = S \otimes_k B$ be the induced Cohen-Macaulay map. Suppose \mathcal{N} is in $\text{MCM}_{\bar{h}} \subseteq \text{mod}_h^{\text{fl}}$ and let $\mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N}$ be an $\text{MCM}_{\bar{h}}$ -approximation of \mathcal{N} . Assume $\text{ob}(T/(JT)^2 \rightarrow \bar{T}, \mathcal{N}) = 0$. Then

$$\nabla_{\bar{T}}(\mathcal{N}) = \nabla_T(\mathcal{M}) \cap \text{Supp } \bar{T}.$$

Proof. By Proposition 9.2 there is a lifting $\mathcal{N}_1 \rightarrow \mathcal{N}$ of \mathcal{N} to $T_1 = T/(JT)^2$. It induces liftings $\mathcal{N}_1(t) \rightarrow \mathcal{N}(t)$ for all k -points $t = (s, \mathfrak{m})$ in $\text{Supp } \bar{T}$. The inclusion $\bar{\tau} : \text{Ext}_{B_{\mathfrak{m}}}^1(\mathcal{N}(t), \mathcal{N}(t)) \rightarrow \text{Ext}_{A_{\mathfrak{m}}}^1(\mathcal{M}(t), \mathcal{M}(t))$ in (11.5.1) commutes with the local Kodaira-Spencer maps. Proceed as in the proof of Corollary 12.8. \square

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COHEN-MACAULAY APPROXIMATION IN FIBRED CATEGORIES

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ABSTRACT. We extend the Auslander-Buchweitz axioms and prove Cohen-Macaulay approximation results for fibred categories. We show that these axioms apply for the fibred category of pairs consisting of a finite type flat family of Cohen-Macaulay rings and modules. In particular such a pair admits an approximation with a flat family of maximal Cohen-Macaulay modules and a hull with a flat family of modules with finite injective dimension. The existence of minimal approximations and hulls in the local, flat case implies extension of upper semi-continuous invariants. As an example of MCM approximation we define a relative version of Auslander's fundamental module.

In the second part we study the induced maps of deformation functors and deduce properties like smoothness and injectivity under general, mainly cohomological conditions on the module. We also provide deformation theory for pairs (algebra, module), e.g. a cohomology for such pairs, a long exact sequence linking this cohomology to the André-Quillen cohomology of the algebra and the Ext cohomology of the module, Kodaira-Spencer classes and maps including a secondary Kodaira-Spencer class, and existence of a versal family for pairs with isolated singularity.

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1. INTRODUCTION

Axiomatic Cohen-Macaulay approximation was introduced by M. Auslander and R.-O. Buchweitz in [7]. We define this theory in terms of fibred categories and obtain approximation results for various classes of flat families of modules.

Let A be a Cohen-Macaulay ring of finite Krull dimension with a canonical module ω_A . Let MCM_A and FID_A denote the categories of maximal Cohen-Macaulay modules and of finite modules with finite injective dimension, respectively. M. Auslander and R.-O. Buchweitz proved in [7] that for any finite A -module N there exists short exact sequences

$$(1.0.1) \quad 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N \rightarrow L' \rightarrow M' \rightarrow 0$$

with M and M' in MCM_A and L and L' in FID_A . The maps $M \rightarrow N$ and $N \rightarrow L'$ in (1.0.1) are called a maximal Cohen-Macaulay approximation and a hull of finite injective dimension, respectively, of the module N . The association $N \mapsto X$ for X equal to M, M', L and L' define functors of corresponding stable categories. In this article we study the continuous properties of these functors.

Linear representations provided by (sheaves) of modules and the associated homological algebra plays an important role in algebra and algebraic geometry, e.g. as a means for classification by providing invariants. Finite complexes have particular properties as seen in the Buchsbaum-Eisenbud acyclicity criterion and the intersection theorems of Peskine, Szpiro and Roberts. However, for a non-regular local ring A , the standard homological invariants are given by the (generally) infinite minimal A -free resolutions, of which very little is known. To stay within finite complexes one can enlarge or change the category of resolving objects and Cohen-Macaulay approximation is a structured way of doing this.

Let D_A denote the subcategory $\text{Add}\{\omega_A\}$ of modules D isomorphic to direct summands of the $\omega_A^{\oplus r}$. A part of the approximation result says that all the modules in FID_A have finite resolutions by objects in D_A . In particular the MCM approximation in (1.0.1) can be extended to a finite resolution

$$(1.0.2) \quad 0 \rightarrow D^{-n} \rightarrow D^{-n+1} \rightarrow \dots \rightarrow D^{-1} \rightarrow M \rightarrow N \rightarrow 0$$

with the D^i in D_A . In the case A is Gorenstein, D_A equals the category of finite projective modules P_A . This generalises: By a result of R. Y. Sharp [42] the functor $\text{Hom}_A(\omega_A, -)$ gives an exact equivalence $\text{D}_A \simeq \text{P}_A$, hence a finite projective resolution is associated to N . In the case A is local, the approximations and the complex can be chosen to be minimal and unique (with $D^i \cong \omega_A^{\oplus d^i}$) and in particular the d^i are invariants of N .

The developments since Auslander and Buchweitz' fundamental work [7] has included studies of invariants defined by Cohen-Macaulay approximation; [14, 8, 25] among several, 'injectivity' and 'surjectivity' properties of the approximation maps; [32, 46, 33], and characterisations of quasi-homogeneous isolated singularities; cf. [27, 36], all exclusively in the Gorenstein case. Noteworthy is [43] where A.-M. Simon and J. R. Strooker related some of these invariants with Hochster's Canonical Element Conjecture and the Monomial Conjecture. In particular these conjectures are equivalent to the vanishing of the δ -invariant of certain cyclic modules over all Gorenstein rings. S. P. Dutta applied the existence of a FID hull to prove a relationship between two of the Serre conjectures on intersection numbers: Failure of vanishing implies failure of higher non-negativity in the Gorenstein case under certain conditions, see [15].

Buchweitz' unpublished manuscript [11], a precursor to [7], contains homological ideas which have influenced subsequent developments (e.g. [35]). Auslander and I.

Reiten elaborated in [9] on [7], mainly with a view towards artin algebras, instigating several generalisations and analogies to Cohen-Macaulay approximation.

However, the ‘relative’ and continuous aspects have received surprisingly little attention. M. Hashimoto has given several new examples of Cohen-Macaulay approximation [24]. In [24, IV 1.4.12] an affine algebraic group G acts on a positively graded Cohen-Macaulay ring T which is flat over a regular base ring R . Hashimoto considers graded maximal Cohen-Macaulay T -modules (which automatically are R -flat) and graded modules locally of finite injective dimension (not R -flat in general), all with G -action. His result (with trivial group) is hence different from our Theorem 5.1. We also note some explicit 1-parameter families of indecomposable finite length modules N_t (for many Gorenstein rings) such that the minimal MCM approximation module M_t is without free summands, see [45].

A central part of the classification problem is to prove the existence of objects with certain properties and to estimate ‘how many’ such objects there are. A natural question is thus whether there is Cohen-Macaulay approximation for flat families of modules. In Theorem 5.1 we give a positive answer to this question. For a Cohen-Macaulay (CM) map $h : S \rightarrow T$ and an S -flat and finite T -module \mathcal{N} there are short exact sequences of S -flat and finite T -modules

$$(1.0.3) \quad 0 \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{N} \rightarrow \mathcal{L}' \rightarrow \mathcal{M}' \rightarrow 0$$

such that the fibres of these sequences give ‘absolute’ approximations and hulls as in the two sequences (1.0.1). Note that T in general is *not* a Cohen-Macaulay ring although the fibres of h are. We consider a category \mathbf{mod}^{fl} of pairs $\xi = (h : S \rightarrow T, \mathcal{N})$ and subcategories MCM, FID and D. They are fibred over the category CM of CM maps and also fibred over the base category of noetherian rings. The approximation and the hull (1.0.3) induces functors of certain quotient categories fibred in additive categories over CM

$$(1.0.4) \quad \mathbf{mod}^{\text{fl}}/\text{D} \rightarrow \text{MCM}/\text{D} \quad \text{and} \quad \mathbf{mod}^{\text{fl}}/\text{D} \rightarrow \text{FID}/\text{D}$$

with analogous properties to the absolute case. If $h : S \rightarrow T$ is a *local* CM map, there is an approximation result with minimal (and hence unique) choices of the two sequences in (1.0.3), see Corollary 5.7 and 6.3.

A major consequence of these results is that any numerical and additive upper semi-continuous invariant of MCM or FID modules by the minimal approximations and hulls induces upper semi-continuous invariants for all finite modules, see Theorem 6.5. Examples of such invariants are given by the ω_A -ranks in the minimal *representing complex* $D^*(N)$ which is an (infinite) extension to the right of the D_A -complex in (1.0.2).

Auslander’s fundamental module E_A for a normal 2-dimensional singularity $\text{Spec } A$ is given by the MCM approximation of the maximal ideal;

$$(1.0.5) \quad 0 \rightarrow \omega_A \rightarrow E_A \rightarrow \mathfrak{m}_A \rightarrow 0$$

which in a certain sense generates all almost split sequences for A , see [6]. As a general example of flat Cohen-Macaulay approximation we define the fundamental module for any finite type CM map of pure relative dimension ≥ 2 , see Corollary 7.3, and more generally a ‘fundamental’ functor of projective modules in Proposition 7.2.

An attractive feature of Auslander and Buchweitz’ theory is its axiomatic formulation with several applications besides the classical case described in the first paragraph, e.g. coherent rings with a cotilting module, the graded case, approximation with modules of Gorenstein dimension 0, and coherent sheaves on a projectively embedded Cohen-Macaulay scheme. See [7] and [24] for more examples. We formulate a relative Cohen-Macaulay approximation theory axiomatically in terms of

categories $\mathbf{D} \subseteq \mathbf{X} \subseteq \mathbf{A}$ fibred in abelian and additive subcategories over a base category \mathbf{C} . In addition to the Auslander-Buchweitz axioms (AB1-4) for the fibre categories we formulate two axioms (BC1-2) regarding base change properties of the fibred categories. AB1-2 and BC1-2 imply the existence of an approximation and a hull which are preserved by any base change, see Theorem 4.4. If AB3 holds too, we get functoriality and adjointness properties in suitable stable categories fibred in additive categories, see Theorem 4.5. In the case described above $\mathbf{C} = \mathbf{CM}$, \mathbf{A} is the category \mathbf{mod} of pairs $(h : S \rightarrow T, \mathcal{N})$ where \mathcal{N} is a finite T -module (no S -flatness) and $\mathbf{X} = \mathbf{MCM}$. Another application of this theory is given in [29].

In the second half of the article we proceed to study properties of continuous families of MCM approximations and FID hulls by homological methods. As a consequence of the existence of minimal approximations and hulls of local flat families there are induced natural maps of deformation functors of pairs of algebra and module:

$$(1.0.6) \quad \mathrm{Def}_{(A,N)} \longrightarrow \mathrm{Def}_{(A,X)} \quad \text{for } X = M, M', L \text{ and } L'$$

There are corresponding maps $\mathrm{Def}_N^A \rightarrow \mathrm{Def}_X^A$ of deformation functors of the modules where A only deforms trivially. Rather weak conditions on N , e.g. grade $N \geq 1$, respectively grade $N \geq 2$, imply the injectivity and formal smoothness of these maps for $X = L'$. If, in addition, there is a versal family in $\mathrm{Def}_{(A,N)}$ (or Def_N^A) then the maps are smooth for the appropriate category of henselian rings, see Theorem 9.2 and Corollary 9.5. As a consequence each CM algebraic k -algebra A with $A/\mathfrak{m}_A \cong k$ and $\dim A \geq 2$ has a finite A -module Q' of finite *projective* dimension with a *universal* deformation in $\mathrm{Def}_{Q'}^A(A)$, see Corollary 9.8. There are analogous general results for $X = M$, see Theorem 9.4 and Corollary 9.6, with applications in Corollary 9.9 and 10.5. E.g. if there is a closed subscheme Z in $\mathrm{Spec} A$ containing the singular locus and with complement U such that $\tilde{N}_U = 0$ and $\mathrm{depth}_Z N \geq 2$ then $\sigma_M : \mathrm{Def}_{(A,N)} \rightarrow \mathrm{Def}_{(A,M)}$ is formally smooth. Or if $\mathrm{Spec} A$ is a 2-dimensional normal Gorenstein singularity and N is torsion-free then the map σ_M is smooth. In this case both functors have versal elements by Theorem 10.2.

Consider a quotient ring $B = A/I$ defined by a regular sequence $I = (f_1, \dots, f_n)$ and an MCM B -module N . Then N is also an A -module with an MCM approximation $M \rightarrow N$. If N has a *lifting* to A/I^2 , then the composition of natural maps $\mathrm{Def}_N^B \rightarrow \mathrm{Def}_M^A \rightarrow \mathrm{Def}_M^A$ is injective, see Theorem 11.1. It turns out that the lifting condition is equivalent to the splitting of $B \otimes_A M \rightarrow N$ (this generalises [8, 4.5]).

The second part of the article also contains some general deformation theory of a pair $(h : S \rightarrow T, \mathcal{N})$ of an algebra and a T -module. We define the graded algebra $\Gamma := T \oplus \mathcal{N}$ and consider the graded André-Quillen cohomology ${}_0\mathrm{H}^*(S, \Gamma, J)$ which govern the obstruction theory of the pair. In the case the graded Γ -module J is concentrated in degree 0 and 1 there is by Proposition 8.8 a natural long-exact sequence which in the case $J = \Gamma$ (with ${}_0\mathrm{H}^*(S, \Gamma) = {}_0\mathrm{H}^*(S, \Gamma, \Gamma)$) gives the suggestive

$$(1.0.7) \quad \begin{aligned} 0 \rightarrow \mathrm{End}_T(\mathcal{N}) \rightarrow {}_0\mathrm{Der}_S(\Gamma) \rightarrow \mathrm{Der}_S(T) \rightarrow \mathrm{Ext}_T^1(\mathcal{N}, \mathcal{N}) \rightarrow {}_0\mathrm{H}^1(S, \Gamma) \rightarrow \\ \mathrm{H}^1(S, T) \rightarrow \mathrm{Ext}_T^2(\mathcal{N}, \mathcal{N}) \rightarrow {}_0\mathrm{H}^2(S, \Gamma) \rightarrow \mathrm{H}^2(S, T) \rightarrow \dots \end{aligned}$$

It relates the cohomology of the pair with the cohomology groups governing the obstruction theory of the algebra T and of the module \mathcal{N} . The sequence is used in the proof of the existence of a versal element in $\mathrm{Def}_{(A,N)}$ where $\mathrm{Spec} A$ is an isolated equidimensional singularity and N is locally free on the smooth locus, see Theorem 10.2. It is also used to define and study the Kodaira-Spencer class $\kappa(\Gamma/S/A)$ in ${}_0\mathrm{H}^1(S, \Gamma, \Omega_{S/A} \otimes \Gamma)$ (where $\Lambda \rightarrow S$ is another ring homomorphism) which maps to the ungraded Kodaira-Spencer class $\kappa(T/S/A)$. In the case the latter is zero

we define a ‘secondary’ Kodaira-Spencer class $\kappa(\sigma, \mathcal{N})$ in $\text{Ext}_T^1(\mathcal{N}, \Omega_{S/A} \otimes \mathcal{N})$ which depends on a choice of an S -algebra splitting σ . This enables us to define ‘global’ Kodaira-Spencer maps

$$(1.0.8) \quad g^F : \text{Der}_A(S) \rightarrow {}_0H^1(S, F, F) \quad \text{and} \quad g^{(\sigma, \mathcal{N})} : \text{Der}_A(S) \rightarrow \text{Ext}_T^1(\mathcal{N}, \mathcal{N}).$$

We also describe how classes and maps are related to the Atiyah class $\text{at}_{T/A}(\mathcal{N})$ in $\text{Ext}_T^1(\mathcal{N}, \Omega_{T/A} \otimes \mathcal{N})$. These results might have a certain independent interest. The arguments are general and should be extendable to the setting of L. Illusie’s [30].

Injectivity of the corresponding local Kodaira-Spencer maps gives a criterion for a global flat family to be non-trivial. The Kodaira-Spencer maps commute with Cohen-Macaulay approximation and this is applied to show that injectivity of the Kodaira-Spencer map is preserved by Cohen-Macaulay approximation under conditions as in Theorem 9.2, 9.4 and 11.1.

To make the text more reader friendly we have included some background material, e.g. on Cohen-Macaulay approximation and Kodaira-Spencer maps, and some of the central technical tools such as a general ‘cohomology and base change’ result and some language of fibred categories. Many results have analogous parts with similar arguments and the policy has been to give a fairly detailed proof of one case and leave the other cases to the reader.

2. PRELIMINARIES

All rings are commutative. If A is a ring, Mod_A denotes the category of A -modules and mod_A denotes the full subcategory of finite A -modules. If A is local then \mathfrak{m}_A denotes the maximal ideal. Subcategories are usually full and essential.

2.1. Axiomatic Cohen-Macaulay approximation. We briefly recall some of the main features of Cohen-Macaulay approximation as introduced by Auslander and Buchweitz in [7]. In this section let \mathbf{A} be an abelian category and $\mathbf{D} \subseteq \mathbf{X} \subseteq \mathbf{A}$ additive subcategories. Let $\hat{\mathbf{X}}$ denote the subcategory of \mathbf{A} of objects N which have finite resolutions $0 \rightarrow M_n \rightarrow \dots \rightarrow M_0 \rightarrow N \rightarrow 0$ with the M_i in \mathbf{X} . If n is the smallest such number, then $\mathbf{X}\text{-res.dim } N = n$. Let $\mathbf{X}\text{-inj.dim } N$ be the minimal n (possibly ∞) such that $\text{Ext}_{\mathbf{A}}^i(M, N) = 0$ for all $i > n$ and all M in \mathbf{X} . Let \mathbf{X}^\perp denote the subcategory of objects L in \mathbf{A} with $\mathbf{X}\text{-inj.dim } L = 0$; the right complement of \mathbf{X} . The left complement ${}^\perp\mathbf{X}$ is defined analogously.

Let N be an object in \mathbf{A} . An \mathbf{X} -approximation and a $\hat{\mathbf{D}}$ -hull of N are exact sequences as in (1.0.1) with L, L' in $\hat{\mathbf{D}}$ and M, M' in \mathbf{X} .

In general any $f : M \rightarrow N$ in \mathbf{A} is called a *right \mathbf{X} -approximation of N* if M is in \mathbf{X} and any $f' : M' \rightarrow N$ with M' in \mathbf{X} factorises through f . Dually, $g : N \rightarrow L$ is called a *left \mathbf{X} -approximation of N* if L is in \mathbf{X} and any $g' : N \rightarrow L'$ with L' in \mathbf{X} factorises through g .

Consider the following conditions on the triple of categories $(\mathbf{A}, \mathbf{X}, \mathbf{D})$.

- (AB1) \mathbf{X} is exact in \mathbf{A} (\mathbf{X} is closed under direct summands and extensions).
- (AB2) \mathbf{D} is a *cogenerator* for \mathbf{X} , i.e. for each object M in \mathbf{X} there is an object D in \mathbf{D} and a short exact sequence $M \rightarrow D \rightarrow M'$ with M' in \mathbf{X} .
- (AB3) \mathbf{D} is \mathbf{X} -injective, i.e. $\mathbf{D} \subseteq \mathbf{X}^\perp$.
- (AB4) \mathbf{A} -epimorphisms in \mathbf{X} are admissible (i.e. their kernels are contained in \mathbf{X}).

If AB1 and AB2, there exist \mathbf{X} -approximations and $\hat{\mathbf{D}}$ -hulls for all objects in $\hat{\mathbf{X}}$ [7, 1.1]. Assume AB1-3. Then any \mathbf{X} -approximation is a right \mathbf{X} -approximation and any $\hat{\mathbf{D}}$ -hull is a left $\hat{\mathbf{D}}$ -approximation. An \mathbf{X} -approximation determines a $\hat{\mathbf{D}}$ -hull and vice versa through the following diagram of short exact sequences; the upper horizontal and right vertical being an \mathbf{X} -approximation and a $\hat{\mathbf{D}}$ -hull of N , D is in

D. The boxed square is (co)cartesian (see [7, 1.4]):

$$(2.0.9) \quad \begin{array}{ccccc} L & \longrightarrow & M & \longrightarrow & N \\ \parallel & & \downarrow & \square & \downarrow \\ L & \longrightarrow & D & \longrightarrow & L' \\ & & \downarrow & & \downarrow \\ & & M' & \xlongequal{\quad} & M' \end{array}$$

Moreover, the category \mathbf{D} is determined by $\mathbf{X} \subset \mathbf{A}$. Indeed $\mathbf{D} = \mathbf{X} \cap \mathbf{X}^\perp$. By [7, 3.9] monomorphisms in $\hat{\mathbf{D}}$ are admissible and $\hat{\mathbf{D}} = \hat{\mathbf{X}} \cap \mathbf{X}^\perp$. Also $\mathbf{X} = {}^\perp \hat{\mathbf{D}} \cap \hat{\mathbf{X}} = {}^\perp \mathbf{D} \cap \hat{\mathbf{X}}$. If \mathbf{X}/\mathbf{D} denotes the quotient category, the \mathbf{X} -approximation induces a right adjoint to the inclusion functor $\mathbf{X}/\mathbf{D} \subseteq \hat{\mathbf{X}}/\mathbf{D}$ and the $\hat{\mathbf{D}}$ -hull induces a left adjoint to the inclusion functor $\hat{\mathbf{D}}/\mathbf{D} \subseteq \hat{\mathbf{X}}/\mathbf{D}$, see [7, 2.8].

A morphism $f : M \rightarrow N$ in \mathbf{A} is called *right minimal* if for any $g : M \rightarrow M$ with $fg = f$ it follows that g is an automorphism. Dually, f is called *left minimal* if for any $h : N \rightarrow N$ with $hf = f$ it follows that h is an automorphism. Note that if $f : M \rightarrow N$ and $f : M' \rightarrow N$ both are right minimal then there exists an isomorphism $g : M \rightarrow M'$ with $f = f'g$, and similarly for left minimal morphisms.

We will simply call an \mathbf{X} -approximation (a $\hat{\mathbf{D}}$ -hull) for minimal if it is right (left) minimal.

Example 2.1. Suppose A is a Cohen-Macaulay ring which possesses a canonical module ω_A in the sense that any localisation in a maximal ideal gives a maximal Cohen-Macaulay module of finite injective dimension and Cohen-Macaulay type 1, cf. [10, 3.3.16]. Let \mathbf{MCM}_A denote the category of maximal Cohen-Macaulay (MCM) A -modules and put $\mathbf{D}_A := \text{Add}\{\omega_A\}$. Then the triple $(\mathbf{A}, \mathbf{X}, \mathbf{D}) = (\mathbf{mod}_A, \mathbf{MCM}_A, \mathbf{D}_A)$ satisfies properties AB1-4, cf. [24, I 4.10.11] and $\hat{\mathbf{X}} = \mathbf{mod}_A$. If A in addition is a local ring, then the \mathbf{MCM}_A -approximation and the $\hat{\mathbf{D}}_A$ -hull can be chosen to be minimal, cf. [43, Sec. 3] or Corollary 6.3.

Let \mathbf{FID}_A^l denote the subcategory of finite A -modules E which have locally finite injective dimension, i.e. $\text{inj.dim}_{A_{\mathfrak{p}}} E_{\mathfrak{p}} < \infty$ for all $\mathfrak{p} \in \text{Spec } A$. The approximation result implies that $\mathbf{FID}_A^l = \hat{\mathbf{D}}_A$: Let L be in $\hat{\mathbf{D}}_A$. By induction on \mathbf{D}_A -res.dim L L is in \mathbf{FID}_A^l . Conversely let E be in \mathbf{FID}_A^l . If $L \rightarrow M \rightarrow E$ is an \mathbf{MCM}_A -approximation of E then M also has locally finite injective dimension. Let M^\vee denote $\text{Hom}_A(M, \omega_A)$ and choose a surjection $A^{\oplus n} \rightarrow M^\vee$. Both M^\vee and the kernel M_1 are MCM. Applying $\text{Hom}_A(-, \omega_A)$ gives (by duality theory) the short exact sequence $M \xrightarrow{i} \omega_A^{\oplus n} \rightarrow M_1^\vee$. But i splits since $\text{Ext}_A^1(M_1^\vee, M) = 0$ by [10, 3.3.3] and so M is in \mathbf{D}_A and E is in $\hat{\mathbf{D}}_A$.

2.2. The representing complex. Consider an abelian category \mathbf{A} and additive subcategories $\mathbf{D} \subseteq \mathbf{X} \subseteq \mathbf{A}$. A *DX-resolution* of an object N in \mathbf{A} is a finite resolution ${}^-C^* \twoheadrightarrow N$ with ${}^-C^i \in \mathbf{D}$ for $i < 0$ and ${}^-C^0 \in \mathbf{X}$. If $L := \text{coker}(d^{-2} : {}^-C^{-2} \rightarrow {}^-C^{-1})$, then the short exact sequence $L \rightarrow {}^-C^0 \rightarrow N$ is an \mathbf{X} -approximation. A *$\hat{\mathbf{D}}$ D-coresolution* of N is a coresolution $N \hookrightarrow {}^+C^*(N)$ such that ${}^+C^0 \in \hat{\mathbf{D}}$, ${}^+C^i \in \mathbf{D}$ and $\ker d^i \in \mathbf{X}$ for $i > 0$. If $M' := \ker d^1$ then the short exact sequence $N \rightarrow {}^+C^0 \rightarrow M'$ is a $\hat{\mathbf{D}}$ -hull. Given AB1 and AB2, each N in $\hat{\mathbf{X}}$ has a DX-resolution and a $\hat{\mathbf{D}}$ D-coresolution. Finally, a bounded below \mathbf{D} -complex $D^*(N) : \dots \rightarrow D^{-1} \rightarrow D^0 \rightarrow D^1 \rightarrow \dots$ with $\ker d^i \in \mathbf{X}$ for all $i \geq 0$ and its only non-trivial cohomology in degree zero with $H^0(D^*) \cong N$ is called a *\mathbf{D} -complex representing N* . A representing complex splits into (and is (re)constructed from) a DX-resolution given by $\dots \rightarrow D^{-1} \rightarrow \ker d^0 \twoheadrightarrow H^0(D^*) = N$ and a $\hat{\mathbf{D}}$ D-coresolution $N \hookrightarrow \text{coker } d^{-1} \rightarrow D^1 \rightarrow \dots$ where $N \hookrightarrow \text{coker } d^{-1}$ is induced by $\ker d^0 \twoheadrightarrow D^0$.

Lemma 2.2. Assume $\text{Ext}_A^1(\mathbf{X}, \hat{\mathbf{D}}) = 0$. Suppose $f : N_1 \rightarrow N_2$ is in $\hat{\mathbf{X}}$. Assume $F^*(N_i)$ exists for $i = 1, 2$ where $F^*(N_i)$ denotes one of the complexes $-C^*(N_i)$, $+C^*(N_i)$ or $D^*(N_i)$. Then f can be extended to an arrow of chain complexes $f^* : F^*(N_1) \rightarrow F^*(N_2)$ which is uniquely defined up to homotopy.

Assume AB1-3 for the triple of categories $(\mathbf{A}, \mathbf{X}, \mathbf{D})$. Then $N \mapsto -C^*(N)$, $N \mapsto +C^*(N)$ and $N \mapsto D^*(N)$ induce functors to the homotopy categories of chain complexes as follows:

$$-C^* : \hat{\mathbf{X}} \rightarrow \mathbf{K}^b(\mathbf{X}) \quad +C^* : \hat{\mathbf{X}} \rightarrow \mathbf{K}^+(\hat{\mathbf{D}}) \quad D^* : \hat{\mathbf{X}}/\mathbf{D} \rightarrow \mathbf{K}^+(\mathbf{D})$$

Proof. The proof for $-C^*(N)$ and $+C^*(N)$ follows standard lines for constructing chain maps and homotopies. The assumption $\text{Ext}_A^1(\mathbf{X}, \hat{\mathbf{D}}) = 0$ is used every time a lifting or extension of an arrow is required.

Let $(D_i^*, d_i^*) = D^*(N_i)$ and let $M_i = \ker d_i^0$ and $L_i = \text{im } d_i^{-1}$. Then there are short exact sequences $L_i \rightarrow M_i \rightarrow N_i$ which by assumption are \mathbf{X} -approximations. Since $\text{Ext}_A^1(M_1, L_2) = 0$, the arrow $N_1 \rightarrow N_2$ extends to the \mathbf{X} -approximation and further on to the negative part of the complexes. If $M'_i = \ker d_i^1$ then the M'_i are in \mathbf{X} by assumption and there are short exact sequences $M_i \rightarrow D_i^0 \rightarrow M'_i$. There is an extension of $M_1 \rightarrow D_2^0$ to $D_1^0 \rightarrow D_2^0$ and an induced arrow $M'_1 \rightarrow M'_2$ which again extends and so on to a chain map $f^* : D_1^* \rightarrow D_2^*$.

Let $g^* : D_1^* \rightarrow D_2^*$ be a chain map, put $g = H^0(g^*)$, $s = f - g$ and $s^* = f^* - g^*$. Suppose s factors through D in \mathbf{D} ; $s = ab$ with $a : D \rightarrow N_2$. Since $\text{Ext}_A^1(D, L_2) = 0$ there exist a lifting $\tilde{a} : D \rightarrow M_2$ of a . Put $h_N = \tilde{a}b$ and continue similarly to construct a homotopy h for the extended negative part:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & D_1^{-1} & \longrightarrow & M_1 & \longrightarrow & N_1 \longrightarrow 0 \\ & \searrow h^{-1} & \downarrow s^{-1} & \swarrow h_M & \downarrow Z^0(s^*) & \swarrow h_N & \downarrow s \\ \cdots & \longrightarrow & D_2^{-2} & \longrightarrow & D_2^{-1} & \longrightarrow & M_2 \longrightarrow N_2 \longrightarrow 0 \end{array}$$

In particular $h_M : M_1 \rightarrow D_2^{-1}$ can be extended to an $h^0 : D_1^0 \rightarrow D_2^{-1}$ with $s^{-1} = h^0 d_1^{-1} + d_2^{-2} h^{-1}$. The construction of the h^i for $i > 0$ is standard. \square

Lemma 2.3. Assume AB1-3 for the triple of categories $(\mathbf{A}, \mathbf{X}, \mathbf{D})$. Given an exact sequence $\varepsilon : 0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$ with objects in $\hat{\mathbf{X}}$. Then there are exact sequences of complexes where ε equals the cohomology:

- (i) $0 \rightarrow -C^*(N_1) \rightarrow -C^*(N_2) \rightarrow -C^*(N_3) \rightarrow 0$
- (ii) $0 \rightarrow +C^*(N_1) \rightarrow +C^*(N_2) \rightarrow +C^*(N_3) \rightarrow 0$
- (iii) $0 \rightarrow D^*(N_1) \rightarrow D^*(N_2) \rightarrow D^*(N_3) \rightarrow 0$ (termwise split exact)

Proof. Choose \mathbf{X} -approximations $L_i \rightarrow M_i \rightarrow N_i$ for $i = 1, 3$. There is an 3×3 commutative diagram of 6 short exact sequences which extends the “horseshoe” diagram, cf. [24, 1.12.11]. One obtains an \mathbf{X} -approximation of N_2 and short exact sequences $m : M_1 \rightarrow M_2 \rightarrow M_3$ and $L_1 \rightarrow L_2 \rightarrow L_3$ in \mathbf{X} and $\hat{\mathbf{D}}$ respectively since both categories are closed by extensions (by AB1 and [7, 3.8]). If $D_i^*[1] \xrightarrow{\eta_i} L_i$ are finite \mathbf{D} -resolutions then since $\text{Ext}_A^1(D_3^{-1}, L_1) = 0$ there is a lifting $\tilde{\eta}_3 : D_3^{-1} \rightarrow L_2$ of η_3 which combined with η_1 gives $\eta_2 : D_1^{-1} \coprod D_3^{-1} \rightarrow L_2$. The kernels of the resulting arrows between short exact sequences give a short exact sequence of objects in $\hat{\mathbf{D}}(S)$. The argument is repeated. Splicing with m in degree zero the short exact sequence of $-C^*$ -resolutions in (i) is obtained.

Choose short exact sequences $M_i \rightarrow D_i^0 \rightarrow M'_i$ for $i = 1, 3$ as in AB2. Since $\text{Ext}_A^1(M_3, D_1^0) = 0$ there is an extension to an arrow of short exact sequences from m to $D_1^0 \rightarrow D_2^0 \rightarrow D_3^0$ with $D_2^0 = D_1^0 \coprod D_3^0$ and $M'_2 := \text{coker}(M_2 \rightarrow D_2^0) \in \mathbf{X}$ by AB1. Repeated application of this argument gives a short exact sequence of

D-coresolutions and splicing with the sequences in (i) gives (iii). Pushout of $M_i \rightarrow D_i^0 \rightarrow M'_i$ along $M_i \rightarrow N_i$ gives a short exact sequence of \hat{D} -hulls and splicing with $D_i^1 \rightarrow D_i^2 \rightarrow \dots$ gives (ii). \square

2.3. Base change. The main tool for reducing properties to the fibres in a flat family will be the base change theorem. We follow the quite elementary and general approach of A. Ogus and G. Bergman [39].

Definition 2.4. Let $h : S \rightarrow T$ be a ring homomorphism and I an S -module. Let F be an S -linear functor of some additive subcategory of Mod_S to Mod_T . Then the *exchange map* e_I for F is defined as the T -linear map $e_I : F(S) \otimes_S I \rightarrow F(I)$ given by $\xi \otimes u \mapsto F(u)(\xi)$ where we consider u as the multiplication map $u : S \rightarrow I$. Let $\mathbf{m}\text{-Spec } T$ denote the set of closed points in $\text{Spec } T$.

Proposition 2.5. *Let $h : S \rightarrow T$ be a ring homomorphism with S noetherian. Suppose $\{F^q : \text{mod}_S \rightarrow \text{mod}_T\}_{q \geq 0}$ is an h -linear cohomological δ -functor.*

- (i) *If the exchange map $e_{S/\mathfrak{n}}^q : F^q(S) \otimes_S S/\mathfrak{n} \rightarrow F^q(S/\mathfrak{n})$ is surjective for all \mathfrak{n} in $Z = \text{im}\{\mathbf{m}\text{-Spec } T \rightarrow \text{Spec } S\}$, then $e_I^q : F^q(S) \otimes_S I \rightarrow F^q(I)$ is an isomorphism for all I in mod_S .*
- (ii) *If $e_{S/\mathfrak{n}}^q$ is surjective for all \mathfrak{n} in Z , then e_I^{q-1} is an isomorphism for all I in mod_S if and only if $F^q(S)$ is S -flat.*

Note that if the F^q in addition extend to functors of all S -modules $F^q : \text{Mod}_S \rightarrow \text{Mod}_T$ which commute with direct limits, then the conclusions are valid for all I in Mod_S .

Example 2.6. Suppose S and T are noetherian. Let $K^* : K^0 \rightarrow K^1 \rightarrow \dots$ be a complex of S -flat and finite T -modules. Define $F^q : \text{mod}_S \rightarrow \text{mod}_T$ by $F^q(I) = H^q(K^* \otimes_S I)$. Then $\{F^q\}_{q \geq 0}$ is an h -linear cohomological δ -functor which extends to all S -modules and commutes with direct limits.

Example 2.7. Suppose S and T are noetherian. Let M and N be finite T -modules with N S -flat. Then the functors $F^q : \text{mod}_S \rightarrow \text{mod}_T$ defined by $F^q(I) = \text{Ext}_T^q(M, N \otimes_S I)$ for $q \geq 0$ give an h -linear cohomological δ -functor which extends to all S -modules and commutes with direct limits.

Let $S \rightarrow T$ and $S \rightarrow S'$ be ring homomorphisms, M a T -module, $T' = T \otimes_S S'$ and N' a T' -module. Then there is a change of rings spectral sequence

$$(2.7.1) \quad E_2^{p,q} = \text{Ext}_{T'}^q(\text{Tor}_p^S(M, S'), N') \Rightarrow \text{Ext}_T^{p+q}(M, N')$$

which, in addition to the isomorphism $\text{Hom}_{T'}(M \otimes_S S', N') \cong \text{Hom}_T(M, N')$, gives edge maps $\text{Ext}_{T'}^q(M \otimes_S S', N') \rightarrow \text{Ext}_T^q(M, N')$ for $q > 0$ which are isomorphisms too if M (or S') is S -flat. If I' is an S' -module we can compose the exchange map $e_{I'}^q$ (regarding I' as S -module) with the inverse of this edge map for $N' = N \otimes_S I'$ and obtain a map $c_{I'}^q$ of T' -modules

$$(2.7.2) \quad c_{I'}^q : \text{Ext}_T^q(M, N) \otimes_S I' \rightarrow \text{Ext}_{T'}^q(M \otimes_S S', N \otimes_S I').$$

Remark 2.8. This is the base change map (in the affine case) considered by A. Altman and S. Kleiman, their conditions are slightly different, see [1, 1.9].

We will use the following geometric notation. Suppose $h : S \rightarrow T$ is a ring homomorphism, M is a T -module and s is a point in $\text{Spec } S$ with residue field $k(s)$. Then M_s denotes the fibre $M \otimes_S k(s)$ of M at s with its natural $T_s = T \otimes_S k(s)$ -module structure. Now Proposition 2.5 implies the following:

Corollary 2.9. *Suppose $S \rightarrow T$ and $S \rightarrow S'$ are homomorphisms of noetherian rings, M and N are finite T -modules, $Z = \text{im}\{\mathbf{m}\text{-Spec } T \rightarrow \text{Spec } S\}$ and q is an integer. Assume that M (if $q > 0$) and N are S -flat.*

- (i) If $\text{Ext}_{T_s}^{q+1}(M_s, N_s) = 0$ for all s in Z , then c_I^q in (2.7.2) is an isomorphism for all S' -modules I' .
- (ii) If in addition $\text{Ext}_{T_s}^{q-1}(M_s, N_s) = 0$ for all $s \in Z$, then $\text{Ext}_T^q(M, N)$ is S -flat.

3. CATEGORIES FIBRED IN ADDITIVE CATEGORIES

We will phrase our results in the language of fibred categories¹. We therefore briefly recall some of the basic notions, taken mainly from A. Vistoli's article in [17]. Then we define quotients of categories fibred in additive categories.

Consider a category \mathbf{C} . Given a category over \mathbf{C} , i.e. a functor $p : \mathbf{F} \rightarrow \mathbf{C}$. To an object T in \mathbf{C} , let $\mathbf{F}(T)$; the *fiber of \mathbf{F} over T* , denote the subcategory of arrows φ in \mathbf{F} such that $p(\varphi) = \text{id}_T$. An arrow $\varphi_1 : \xi \rightarrow \xi_1$ in \mathbf{F} is *cocartesian* if for any arrow $\varphi_2 : \xi \rightarrow \xi_2$ in \mathbf{F} and any arrow $f_{21} : p(\xi_1) \rightarrow p(\xi_2)$ in \mathbf{C} with $f_{21}p(\varphi_1) = p(\varphi_2)$ there exists a unique arrow $\varphi_{21} : \xi_1 \rightarrow \xi_2$ with $p(\varphi_{21}) = f_{21}$ and $\varphi_{21}\varphi_1 = \varphi_2$. If for any arrow $f : T \rightarrow T'$ in \mathbf{C} and any object ξ in \mathbf{F} with $p(\xi) = T$ there exists a cocartesian arrow $\varphi : \xi \rightarrow \xi'$ for some ξ' with $p(\varphi) = f$, then \mathbf{F} (or rather $p : \mathbf{F} \rightarrow \mathbf{C}$) is a *fibred category*. Moreover, ξ' will be called a *base change* of ξ by f . If ξ'' is another base change of ξ by f then ξ' and ξ'' are isomorphic over T' by a unique isomorphism. We shall also say that a property P of objects in the fibres of \mathbf{F} is *preserved by base change* if $P(\xi)$ implies $P(\xi')$ for any base change ξ' of ξ . A morphism of fibred categories is a functor $F : \mathbf{F}_1 \rightarrow \mathbf{F}_2$ with $p_2 F = p_1$ such that φ cocartesian implies $F(\varphi)$ cocartesian. If F in addition is an inclusion of categories, \mathbf{F}_1 is a fibred subcategory of \mathbf{F}_2 . A category with all arrows being isomorphisms is a groupoid. A fibred category \mathbf{F} over \mathbf{C} is called a *category fibred in groupoids* (often abbreviated to groupoid) if all fibres $\mathbf{F}(T)$ are groupoids. Then all arrows in \mathbf{F} are cocartesian. If all fibres $\mathbf{F}(T)$ only contain identities, then \mathbf{F} is called a *category fibred in sets*.

Lemma 3.1. *Given functors $F : \mathbf{F} \rightarrow \mathbf{G}$ and $q : \mathbf{G} \rightarrow \mathbf{C}$ and suppose q is fibred in sets. Then F is fibred (in groupoids/sets) if and only if qF is fibred (in groupoids/sets).*

If T is an object in a category \mathbf{C} let \mathbf{C}/T denote the comma category of arrows to T . Then the forgetful functor $\mathbf{C}/T \rightarrow \mathbf{C}$ is fibred in sets. If $p : \mathbf{F} \rightarrow \mathbf{C}$ is fibred (in groupoids/sets), ξ is an object in \mathbf{F} and $T = p(\xi)$, then there is a natural functor $p_\xi : \mathbf{F}/\xi \rightarrow \mathbf{C}/T$. The composition $\mathbf{F}/\xi \rightarrow \mathbf{F} \rightarrow \mathbf{C}$ is clearly fibred (in groupoids/sets) and hence $\mathbf{F}/\xi \rightarrow \mathbf{C}/T$ is fibred (in groupoids/sets) by Lemma 3.1. If $p : \mathbf{F} \rightarrow \mathbf{C}$ is a functor and \mathbf{C}' is a subcategory of \mathbf{C} we can define the *restriction* $p' : \mathbf{F}|_{\mathbf{C}'} \rightarrow \mathbf{C}'$ of \mathbf{F} to \mathbf{C}' by picking for $\mathbf{F}|_{\mathbf{C}'}$ the objects and morphisms in \mathbf{F} that p takes into \mathbf{C}' . It follows that $\mathbf{F}|_{\mathbf{C}'}$ is fibred (in groupoids/sets) if \mathbf{F} is.

The composition of two cocartesian arrows is cocartesian and isomorphisms are cocartesian. Hence the subcategory \mathbf{F}_{coCa} of cocartesian arrows in a fibred category \mathbf{F} over \mathbf{C} is fibred in groupoids. If \mathbf{F} is fibred in groupoids there is an associated category fibred in sets $\bar{\mathbf{F}} \rightarrow \mathbf{C}$ defined by identifying all isomorphic objects in all fibres $\mathbf{F}(T)$ and identifying arrows accordingly. If \mathbf{F} is fibred in sets one defines a functor $F : \mathbf{C} \rightarrow \mathbf{Sets}$ by $F(T) := \mathbf{F}(T)$ and $F(f) : F(T) \rightarrow F(T')$ is defined by $F(f)(\xi) := \eta_{\xi, f}$ where $\varphi_{\xi, f} : \xi \rightarrow \eta_{\xi, f}$ is the (in this case) unique cocartesian lifting of f . From a functor $G : \mathbf{C} \rightarrow \mathbf{Sets}$ one defines a category fibred in sets, and these two operations are inverse up to natural equivalences.

Definition 3.2. An *additive (abelian) category \mathbf{F} over \mathbf{C}* is a functor $p : \mathbf{F} \rightarrow \mathbf{C}$ such that:

¹We have chosen to work with rings instead of (affine) schemes. Our definition of a fibred category $p : \mathbf{F} \rightarrow \mathbf{C}$ reflects this choice and is equivalent to the functor of opposite categories $p^{\text{op}} : \mathbf{F}^{\text{op}} \rightarrow \mathbf{C}^{\text{op}}$ being a fibred category as defined in [17].

- (i) The fibre $F(T)$ is an additive (abelian) category for all objects T in C .
- (ii) For all objects ξ_1 and ξ_2 in F and arrows $f : p(\xi_1) \rightarrow p(\xi_2)$ in C ,

$$\text{Hom}_f(\xi_1, \xi_2) := \{\varphi \in \text{Hom}_F(\xi_1, \xi_2) \mid p(\varphi) = f\}$$

is an abelian group, and composition of arrows

$$\text{Hom}_{f_2}(\xi_2, \xi_3) \times \text{Hom}_{f_1}(\xi_1, \xi_2) \rightarrow \text{Hom}_{f_2 f_1}(\xi_1, \xi_3)$$

is bilinear.

A morphism $F : F_1 \rightarrow F_2$ of additive (abelian) categories over C is a *linear* functor F over C , i.e. which gives linear maps of Hom-groups. If in addition F is an inclusion of categories then F_1 is an additive (abelian) subcategory of F_2 over C . A category F over C is *fibred in additive (abelian) categories*, abbreviated by FAd (FAB), if F is both fibred and additive (abelian) over C . Morphisms should be linear and preserve cocartesian arrows. A FAd subcategory is a morphism of FAds which is an inclusion of categories. For $i = 1, 2$ let A_i be a FAB over C and $X_i \subseteq A_i$ a FAd subcategory such that the fibre categories $X_i(T)$ are exact. Then a morphism of FAds $F : X_1 \rightarrow X_2$ is *exact* if F preserves short exact sequences for all the fibre categories.

Note that in a FAd finite (co)products in the fibres are preserved by base change. Given a FAd subcategory $D \subseteq F$. Two arrows φ_1 and φ_2 in F are D -equivalent if $p(\varphi_1) = p(\varphi_2)$ and $\varphi_1 - \varphi_2$ factors through an object in D . Write $\varphi_1 \sim \varphi_2$. Define the quotient category F/D over C to have the same objects as F and $\text{Hom}_{F/D}(\xi_1, \xi_2) := \text{Hom}_F(\xi_1, \xi_2) / \sim$. The natural map to C makes F/D an additive category over C and the natural functor $F \rightarrow F/D$ is linear over C .

Lemma 3.3. *If $\varphi_1 : \xi \rightarrow \xi_1$ is cocartesian in F and $\varphi : \xi_1 \rightarrow \xi_2$ is any arrow such that $\varphi\varphi_1 \sim 0$ then $\varphi \sim 0$.*

Proof. Suppose $\varphi\varphi_1 = \beta\alpha$ with $\alpha : \xi \rightarrow \delta$ and with δ in D . If $p(\beta) : T' \rightarrow T_2$ then since D is a fibred subcategory there exists an arrow $\delta \rightarrow \delta_2$ which is cocartesian in F and with $p(\delta_2) = p(\xi_2) = T_2$. Replacing δ with δ_2 we assume $p(\delta) = T_2$. Since φ_1 is cocartesian there exists a unique arrow $\tau : \xi_1 \rightarrow \delta$ with $\tau\varphi_1 = \alpha$. Since φ_1 is cocartesian uniqueness implies that $\beta\tau = \varphi$. \square

Lemma 3.4. *Given a FAd subcategory $D \subseteq F$ over C , then the quotient category F/D is FAd over C and the quotient morphism $F \rightarrow F/D$ is a morphism of FAds.*

Proof. We first show that if $\varphi_1 : \xi \rightarrow \xi_1$ is cocartesian in F then its image $[\varphi_1]$ in F/D is cocartesian. Given $\varphi_2 : \xi \rightarrow \xi_2$ and $\theta : \xi_1 \rightarrow \xi_2$ with $\theta\varphi_1 = \varphi_2$. Suppose $\theta' : \xi_1 \rightarrow \xi_2$ with $p(\theta') = p(\theta)$ satisfies $\theta'\varphi_1 \sim \varphi_2$. If $\varphi = \theta' - \theta$ then $\varphi\varphi_1 \sim 0$ so by Lemma 3.3 $\varphi \sim 0$. Now we show that $[\theta]$ is independent of the representations of the other maps. Let $\varphi'_i : \xi \rightarrow \xi_i$ with $\varphi'_i \sim \varphi_i$ and suppose (as we may) that θ' satisfies $\theta'\varphi'_1 = \varphi'_2$ with $p(\theta') = p(\theta)$. Again let $\varphi = \theta' - \theta$. Then $0 \sim \varphi'_2 - \varphi_2 = \theta'\varphi'_1 - \theta\varphi_1 = \theta'(\varphi'_1 - \varphi_1) + \varphi\varphi_1 \sim \varphi\varphi_1$. By Lemma 3.3 $\varphi \sim 0$. Given $f : T \rightarrow T_1$ and ξ in F/D with $p(\xi) = T$ there exists a cocartesian $\varphi_1 : \xi \rightarrow \xi_1$ in F with $p(\varphi_1) = f$ and by what we have done $[\varphi_1]$ is cocartesian in F/D . \square

Note that there are in general more cocartesian arrows in F/D than those in the image of cocartesian arrows in F . The following lemma characterises the cocartesian arrows in the quotient category:

Lemma 3.5. *If ρ and θ are composeable arrows in F with ρ cocartesian and θ inducing an isomorphism in F/D , then $[\theta\rho]$ is cocartesian in F/D . Conversely, suppose $[\varphi] : \xi_1 \rightarrow \xi_2$ is cocartesian in F/D over $f : T_1 \rightarrow T_2$. Then for any base change $\rho : \xi_1 \rightarrow \xi_1^\#$ of ξ_1 over f in F , the induced arrow $\varphi^\# : \xi_1^\# \rightarrow \xi_2$ gives an isomorphism in $F/D(T_2)$.*

Proof. If $\rho : \xi_1 \rightarrow \xi_2$ is cocartesian and $[\theta] : \xi_2 \rightarrow \xi_3$ is an isomorphism, let $\varphi = \theta\rho$. If $\tau : \xi_1 \rightarrow \xi_4$, $p(\xi_i) = T_i$ and there is a map $f : T_3 \rightarrow T_4$ with $p(\tau) = fp(\theta)p(\rho)$, then there is a unique arrow $\mu : \xi_2 \rightarrow \xi_4$ above $fp(\theta)$ with $\mu\rho = \tau$. This gives the arrow $[\mu][\theta]^{-1} : \xi_3 \rightarrow \xi_4$. If $\mu_i : \xi_3 \rightarrow \xi_4$ for $i = 1, 2$ are two arrows with $[\mu_i][\varphi] = [\tau]$, then $[\mu_1][\theta] = [\mu_2][\theta]$ since $[\rho]$ is cocartesian in \mathbf{F}/\mathbf{D} by Lemma 3.4. Since $[\theta]$ is an isomorphism, $[\mu_1] = [\mu_2]$.

Conversely, since $[\varphi]$ is cocartesian there is a unique arrow $[\psi] : \xi_2 \rightarrow \xi_1^\#$ in $\mathbf{F}/\mathbf{D}(T_2)$ with $[\psi\varphi] = [\rho]$. By Lemma 3.4 $[\rho]$ is cocartesian. It follows that $[\varphi^\#] = [\psi]^{-1}$. \square

4. COHEN-MACAULAY APPROXIMATION IN FIBRED CATEGORIES

Given a category \mathbf{C} and a category \mathbf{A} fibred in abelian categories over \mathbf{C} . Base change by an $f : T \rightarrow T'$ in \mathbf{C} applied to the objects in a complex $\dots \rightarrow N_d \rightarrow N_{d-1} \rightarrow \dots$ in $\mathbf{A}(T)$ can by Lemma 3.3 be uniquely extended to a complex and yield a commutative diagram where the vertical arrows are the cocartesian base change arrows:

$$\begin{array}{ccccccc} \dots & \longrightarrow & N_{d+1} & \longrightarrow & N_d & \longrightarrow & N_{d-1} \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & N_{d+1}^\# & \longrightarrow & N_d^\# & \longrightarrow & N_{d-1}^\# \longrightarrow \dots \end{array}$$

Similarly base change of a commutative diagram Δ in $\mathbf{A}(T)$ gives a commutative diagram $\Delta^\#$ and the base change arrows give an arrow of diagrams $\Delta \rightarrow \Delta^\#$.

Let $\mathbf{X} \subseteq \mathbf{A}$ be a FAd subcategory. Consider the following two conditions on the pair (\mathbf{A}, \mathbf{X}) and an object T in \mathbf{C} .

- (BC1) If $\alpha : A_1 \rightarrow A_2$ is an epimorphism in $\mathbf{A}(T)$ and $f : T \rightarrow T'$ is an arrow in \mathbf{C} then any base change of α by f is an epimorphism in $\mathbf{A}(T')$.
- (BC2) Let $\xi : 0 \rightarrow A \rightarrow B \rightarrow M \rightarrow 0$ be an exact sequence in $\mathbf{A}(T)$ with M in $\mathbf{X}(T)$ and $f : T \rightarrow T'$ is an arrow in \mathbf{C} . Then any base change of ξ by f is an exact sequence in $\mathbf{A}(T')$.

The first condition would be satisfied if base change had a right adjoint. The second condition mimics flatness for all objects in $\mathbf{X}(T)$.

The following is an elementary, but essential technical consequence of BC1.

Lemma 4.1. *Let \mathbf{A} be a category fibred in abelian categories over \mathbf{C} which satisfies BC1 for T in \mathbf{C} . Let $c : \dots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \dots$ be an acyclic complex in $\mathbf{A}(T)$ which remains exact after a base change $\dots \rightarrow L_n^\# \rightarrow L_{n-1}^\# \rightarrow \dots$ of c by $f : T \rightarrow T'$. Then base change of $K_n := \ker\{d_{n-1} : L_{n-1} \rightarrow L_{n-2}\}$ by f is isomorphic to $\ker d_{n-1}^\#$ for all n .*

Proof. Let $Q_n = \ker d_{n-1}^\#$. Since the composition $K_n^\# \rightarrow L_{n-1}^\# \rightarrow L_{n-2}^\#$ by Lemma 3.3 is zero (as $K_n \rightarrow K_n^\#$ is cocartesian), there is a factorisation $\rho : K_n^\# \rightarrow Q_n$ of $K_n^\# \rightarrow L_{n-1}^\#$. On the other hand the composition $L_{n+1}^\# \rightarrow L_n^\# \rightarrow K_n^\#$ is zero too, hence there is an arrow from $\text{coker } d_{n+1}^\# \cong Q_n$ to $K_n^\#$ which is a section of ρ . By assumption $L_n^\# \rightarrow K_n^\#$ is an epimorphism. It follows that $Q_n \cong K_n^\#$. \square

Definition 4.2. Given FAd subcategories $\mathbf{D} \subseteq \mathbf{X} \subseteq \mathbf{A}$. Let $\hat{\mathbf{X}}^{\text{fl}}(T)$ denote the additive subcategory of $\mathbf{A}(T)$ with objects N which have a finite \mathbf{X} -resolution $M_* \rightarrow N$ which is preserved as resolution by any base change. Let $\hat{\mathbf{X}}^{\text{fl}} \subseteq \mathbf{A}$ denote the resulting FAd subcategory. Let $\hat{\mathbf{D}}^{\text{fl}}(T)$ denote the additive subcategory of $\mathbf{A}(T)$ with objects L which have a $\mathbf{D}(T)$ -resolution $D_* \rightarrow L$ which is preserved as resolution by any base change. Let $\hat{\mathbf{D}}^{\text{fl}} \subseteq \mathbf{A}$ denote the resulting FAd subcategory.

The reasoning in the beginning of this section combined with Lemma 2.2 gives the following.

Lemma 4.3. *Let $\eta : \dots \rightarrow E_n \rightarrow E_{n-1} \rightarrow \dots$ and $\lambda : \dots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots$ be complexes in $\mathbf{A}(T)$ and $\eta^\#$ and $\lambda^\#$ the complexes resulting from base change over $f : T \rightarrow T'$. If η is homotopic to λ then $\eta^\#$ is homotopic to $\lambda^\#$.*

In particular; if N in $\mathbf{A}(T)$ has one DX-resolution ($\hat{\mathbf{D}}\mathbf{D}$ -coresolution) which is preserved by base change then all DX-resolutions ($\hat{\mathbf{D}}\mathbf{D}$ -coresolutions) are preserved by base change.

Theorem 4.4. *Let \mathbf{A} be a category fibred in abelian categories over \mathbf{C} and let $\mathbf{D} \subseteq \mathbf{X} \subseteq \mathbf{A}$ be inclusion morphisms of categories fibred in additive categories. Fix an object T in \mathbf{C} . Assume BC1-2 for (\mathbf{A}, \mathbf{X}) and T , and AB1-2 for the triple of categories $(\mathbf{A}(T), \mathbf{X}(T), \mathbf{D}(T))$. Then any object N in $\hat{\mathbf{X}}^\mathbf{fl}(T)$ admits an $\mathbf{X}(T)$ -approximation and a $\hat{\mathbf{D}}(T)$ -hull;*

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N \rightarrow L' \rightarrow M' \rightarrow 0$$

with M and M' in $\mathbf{X}(T)$ and L and L' in $\hat{\mathbf{D}}^\mathbf{fl}(T)$, which are preserved by any base change.

Proof. The proof is a variation of the original proof of [7, 1.1]. For every N in $\hat{\mathbf{X}}^\mathbf{fl}(T)$ let $r(N)$ denote the minimal length of an $\mathbf{X}(T)$ -resolution $M_* \rightarrow N$ which is preserved by base change. The proof is by induction on $r(N)$. If $r(N) = 0$ then N is in \mathbf{X} and so is its own \mathbf{X} -approximation, while AB2 provides a short exact sequence $N \rightarrow D \rightarrow M'$ which is a $\hat{\mathbf{D}}(T)$ -hull with D in $\mathbf{D}(T) \subseteq \hat{\mathbf{D}}^\mathbf{fl}(T)$. The approximation is trivially preserved by base change, the hull because of BC2. Assume $r = r(N) > 0$ and let $0 \rightarrow M_r \rightarrow \dots \rightarrow M_0 \rightarrow N$ be an $\mathbf{X}(T)$ -resolution of minimal length preserved by base change. Then $N_1 = \ker(M_0 \rightarrow N)$ is in $\hat{\mathbf{X}}^\mathbf{fl}(T)$ by Lemma 4.1 and $r(N_1) = r - 1$. By induction there is a $\hat{\mathbf{D}}(T)$ -hull $N_1 \rightarrow L \rightarrow M'_1$ with L in $\hat{\mathbf{D}}^\mathbf{fl}$ which is preserved by base change. Pushout of $e : N_1 \rightarrow M_0 \rightarrow N$ along $N_1 \rightarrow L$ gives an $\mathbf{X}(T)$ -approximation $L \rightarrow M \rightarrow N$ by AB1. In the commutative diagram obtained by a base change;

$$(4.4.1) \quad \begin{array}{ccccc} N_1^\# & \longrightarrow & M_0^\# & \longrightarrow & N^\# \\ \downarrow & & \downarrow & & \parallel \\ L^\# & \longrightarrow & M^\# & \longrightarrow & N^\# \\ \downarrow & & \downarrow & & \\ (M'_1)^\# & \xlongequal{\quad} & (M'_1)^\# & & \end{array}$$

the upper row (by Lemma 4.1) and the columns (by BC2) are short exact sequences. It follows that the middle row is a short exact sequence.

By AB2 there is a short exact sequence $M \rightarrow D \rightarrow M'$ with D in $\mathbf{D}(T)$ and M' in $\mathbf{X}(T)$. Pushout of $M \rightarrow D \rightarrow M'$ along $M \rightarrow N$ gives a short exact sequence $h : N \rightarrow L' \rightarrow M'$. Since the induced sequence $L \rightarrow D \rightarrow L'$ is short exact, L' is contained in $\hat{\mathbf{D}}(T)$. Applying a base change we obtain the following commutative diagram:

$$(4.4.2) \quad \begin{array}{ccccc} L^\# & \longrightarrow & M^\# & \longrightarrow & N^\# \\ \parallel & & \downarrow & & \downarrow \\ L^\# & \longrightarrow & D^\# & \longrightarrow & (L')^\# \\ & & \downarrow & & \downarrow \\ & & (M')^\# & \xlongequal{\quad} & (M')^\# \end{array}$$

The upper row and (by BC2) the two columns are short exact sequences. It follows that the middle row is a short exact sequence and hence that L' is contained in \hat{D}^{fl} . \square

Sequences as in Theorem 4.4 preserved by any base change will be called an X -approximation and a \hat{D}^{fl} -hull of N respectively.

Lemma 3.4 makes the following definition reasonable. Three categories fibred in additive categories (FAds) A_i , $i = 1, 2, 3$, an inclusion of FAds $A_1 \subseteq A_2$, and a morphism of FAds $F : A_2 \rightarrow A_3$ equivalent to the quotient morphism $A_2 \rightarrow A_2/A_1$ is called a *short exact sequence of categories fibred in additive categories* and is denoted by $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$.

Theorem 4.5. *Let A be a category fibred in abelian categories over C and let $D \subseteq X \subseteq A$ be inclusion morphisms of categories fibred in additive categories. Assume BC1-2 for the pair (A, X) and AB1-3 for the triple of categories $(A(T), X(T), D(T))$, for all objects T in C . Then:*

- (i) *The X -approximation induces a morphism of categories fibred in additive categories $j^! : \hat{X}^{\text{fl}}/D \rightarrow X/D$ which is a right adjoint to the full and faithful inclusion morphism $j_! : X/D \rightarrow \hat{X}^{\text{fl}}/D$.*
- (ii) *The \hat{D}^{fl} -hull induces a morphism of categories fibred in additive categories $i^* : \hat{X}^{\text{fl}}/D \rightarrow \hat{D}^{\text{fl}}/D$ which is a left adjoint to the full and faithful inclusion morphism $i_* : \hat{D}^{\text{fl}}/D \rightarrow \hat{X}^{\text{fl}}/D$.*
- (iii) *Together these maps give the following commutative diagram of short exact sequences of categories fibred in additive categories:*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \hat{D}^{\text{fl}}/D & \xrightarrow{i_*} & \hat{X}^{\text{fl}}/D & \xrightarrow{j^!} & X/D \longrightarrow 0 \\
 & & \text{id} \downarrow & & \parallel & & \uparrow \text{id} \\
 0 & \longleftarrow & \hat{D}^{\text{fl}}/D & \xleftarrow{i^*} & \hat{X}^{\text{fl}}/D & \xleftarrow{j_!} & X/D \longleftarrow 0
 \end{array}$$

Proof. In each fibre most of these statements are true by the arguments in the proof of [7, 2.8] since we have Theorem 4.4. The general cases are reduced to fibre cases by applying base change. First we have to establish the functors. Note that the quotient categories involved are FAds over C by Lemma 3.4. Let $p : A \rightarrow C$ denote the fibration. For each N_i in \hat{X}^{fl} put $T_i = p(N_i)$ and choose a \hat{D}^{fl} -hull $\iota_i : N_i \rightarrow L_i \rightarrow M_i$ which exists by Theorem 4.4 and such that $\iota_i = \text{id}$ if N_i is in \hat{D}^{fl} . For each arrow $\psi : N_1 \rightarrow N_2$ choose an arrow $\lambda_{21} : L_1 \rightarrow L_2$ commuting with ψ . This arrow is obtained as a composition of a base change $L_1 \rightarrow L_1^{\#}$ over $p(\psi) : T_1 \rightarrow T_2$ with an extension $L_1^{\#} \rightarrow L_2$ of $N_1^{\#} \rightarrow L_2$ obtained since $\text{Ext}_{A(T_2)}^1(M_1^{\#}, L_2) = 0$ by [7, 2.5]. If composeable it follows from [7, 2.8] that $\lambda_{32}\lambda_{21} \sim \lambda_{31}$.

There is a unique arrow $\varphi : N_1^{\#} \rightarrow N_2$ induced by ψ . If $\lambda'_{21} : L_1 \rightarrow L_2$ is an extension of $\psi' : N_1 \rightarrow N_2$ with $p(\psi') = p(\psi)$ such that $\delta_1 := \psi - \psi'$ is equivalent to 0, we have by Lemma 3.3 that $\delta : N_1^{\#} \rightarrow N_2$ induced from δ_1 by base change factors through an object D in $D(T_2)$. It follows that δ factors through $N_1^{\#} \rightarrow L_1^{\#}$. Let τ denote the composition $L_1^{\#} \rightarrow N_2 \rightarrow L_2$ (so $\tau \sim 0$). Let η be a base change over $p(\psi)$ of the difference of the two extensions; $\eta = (\lambda_{21} - \lambda'_{21})^{\#}$. One calculates that $(\eta - \tau)\iota_1^{\#} = 0$, hence $\eta - \tau$ is induced by an arrow $M_1^{\#} \rightarrow L_2$ which lifts to an arrow $M_1^{\#} \rightarrow D^0$ where $D^* \twoheadrightarrow L_2$ is a finite D -resolution of L_2 (since $\text{Ext}_{A(T_2)}^1(M_1^{\#}, \hat{D}(T_2)) = 0$). Hence $\eta - \tau \sim 0$, so $\eta \sim 0$ and $\lambda_{21} \sim \lambda'_{21}$. We have shown that $i^* : \hat{X}^{\text{fl}}/D \rightarrow \hat{D}^{\text{fl}}/D$ is a well defined functor. To show that i^* preserves cocartesian arrows we apply Lemma 3.5: If $[\psi] : N_1 \rightarrow N_2$ is cocartesian in \hat{X}^{fl}/D

then the induced map $[\varphi] : N_1^\# \rightarrow N_2$ is an isomorphism and by [7, 2.8] so is any extension $L_1^\# \rightarrow L_2$ of $[\varphi]$. Composed with the base change $L_1 \rightarrow L_1^\#$ we get a cocartesian arrow in $\hat{D}^\#/\mathcal{D}$ by Lemma 3.5.

A similar argument gives that the morphism $j^! : \hat{X}^\#/\mathcal{D} \rightarrow \mathcal{X}/\mathcal{D}$ induced by (choices of) \mathcal{X} -approximation also is well defined as a map of fibred categories.

To prove adjointness for the pair $(j_!, j^!)$ consider the chosen \mathcal{X} -approximation $L \rightarrow M \xrightarrow{\pi} N$ of N in $\hat{X}^\#(T)$. Given $\varphi_1 : M_1 \rightarrow N$ with M_1 in $\mathcal{X}(T_1)$ and $f = p(\varphi_1)$. Let $\varphi : M_1^\# \rightarrow N$ be induced by a base change of M_1 by f . Since $\text{Ext}_{A(T)}^1(M_1^\#, L) = 0$, φ can be lifted to an arrow $\psi : M_1^\# \rightarrow M$. Composing ψ with the base change $M_1 \rightarrow M_1^\#$ gives a lifting of φ_1 which shows surjectivity of the adjointness map $[\pi \circ -]$. To prove injectivity consider for $i = 2, 3$ arrows $\psi_i : M_1 \rightarrow M$ in \mathcal{X} with $\pi\psi_2 = \pi\psi_3$. Since $p(\pi) = \text{id}$ we have $p(\psi_2) = p(\psi_3) = f$ and we can define $\psi_1 = \psi_2 - \psi_3$ with $\pi\psi_1 \sim 0$. Base change by f induces a $\psi : M_1^\# \rightarrow M$ from ψ_1 . Lemma 3.3 gives $\pi\psi \sim 0$. The argument in [7, 2.8] implies that ψ and hence ψ_1 factors through an object in $\mathcal{D}(T)$. Analogous arguing gives the adjointness of the pair (i^*, i_*) .

The commutativity of the diagram in (iii) follows by definition. For $i^*j_! = 0 = j^!i_*$ see [7, 2.8]. We prove exactness in the upper row. Given $\varphi : N_1 \rightarrow N_2$ in $\hat{X}^\#$ with $f = p(\varphi) : T_1 \rightarrow T_2$ such that $j^![\varphi] = 0$. If $\pi_i : M_i \rightarrow N_i$ are the chosen \mathcal{X} -approximations, $j^![\varphi]$ is represented by a lifting $\psi : M_1 \rightarrow M_2$ and the assumption is that ψ factors through an object D of \mathcal{D} . We claim that φ factors through an object in $\hat{D}^\#$. By base change it's sufficient to prove the special case $f = \text{id}_{T_2}$. If M is any object in $\mathcal{X}(T)$ we have that the composition $\text{Ext}_{A(T)}^1(M, N_1) \cong \text{Ext}_{A(T)}^1(M, M_1) \xrightarrow{\psi_*} \text{Ext}_{A(T)}^1(M, M_2) \cong \text{Ext}_{A(T)}^1(M, N_2)$ is φ_* which hence equals 0. If $e : N_1 \rightarrow L'_1 \rightarrow M'_1$ is a $\hat{D}^\#$ -hull of N_1 , the connecting takes φ to $\varphi_*e \in \text{Ext}_{A(T)}^1(M'_1, N_2)$, i.e. to 0, and so there exists a $\delta : L'_1 \rightarrow N_2$ which induces φ . Exactness in the lower row is analogous. \square

Proposition 4.6. *Let \mathcal{A} be a category fibred in abelian categories over \mathcal{C} and let $\mathcal{D} \subseteq \mathcal{X} \subseteq \mathcal{A}$ be inclusion morphisms of categories fibred in additive categories. Fix an object T in \mathcal{C} . Assume BC1-2 for $(\mathcal{A}, \mathcal{X})$ and T , and AB1-3 for the triple of categories $(\mathcal{A}(T), \mathcal{X}(T), \mathcal{D}(T))$. Then:*

- (i) $\hat{D}^\#(T) = \hat{X}^\#(T) \cap \mathcal{X}(T)^\perp$ and $\mathcal{D}(T) = \mathcal{X}(T) \cap \hat{D}^\#(T)$.
- (ii) $\hat{X}^\#(T)$ and $\hat{D}^\#(T)$ are closed under extensions.
- (iii) Exact sequences $\dots \rightarrow N_n \xrightarrow{d_n} N_{n-1} \rightarrow \dots$ with objects N_i and kernels $\ker d_i$ in $\hat{X}^\#(T)$ remain exact after base change.

If in addition AB4, then:

- (iv) Epimorphisms in $\hat{X}^\#(T)$ are admissible.
- (v) $\text{Add } \hat{X}^\#(T) = \hat{X}^\#(T)$ and $\text{Add } \hat{D}^\#(T) = \hat{D}^\#(T)$.

Proof. For (i) the proofs of [7, 3.6-7] works with the fl-s too by Theorem 4.4. By (i) it's sufficient to prove (ii) for $\hat{X}^\#(T)$. Let $e : N_1 \rightarrow N_2 \rightarrow N_3$ be a short exact sequence in $\hat{X}(T)$. If N_1 and N_3 are in $\hat{X}^\#(T)$, Theorem 4.4 and Lemma 4.3 implies that ${}^\perp C^*(N_1)$ and ${}^\perp C^*(N_3)$ in Lemma 4.3 are preserved as resolutions by base change. Together with BC2 this implies that base change of the short exact sequence of resolutions in Lemma 2.3 (i) gives a short exact sequence of DX-resolutions. For (iii) the long exact sequence is broken into short exact sequences ξ_i with objects in $\hat{X}^\#(T)$. By Lemma 4.1 it is sufficient to prove the claim for short exact sequences. By Lemma 2.3 the short exact sequence of DX-resolutions in

Lemma 2.3 (i) is preserved by base change. It follows that the short exact sequences ξ_i remain exact after base change.

For (iv); if N_2 and N_3 in e are in $\hat{X}^{\text{fl}}(T)$ then N_1 is in $\hat{X}(T)$ by AB4, see [7, 3.5]. The argument proceeds as for extensions.

By (i) it's sufficient to prove (v) for \hat{X}^{fl} . If $N_1 \coprod N_2$ is an object in $\hat{X}^{\text{fl}}(T)$, then N_i is in $\hat{X}(T)$ for $i = 1, 2$ by [7, 3.4]. By Lemma 4.3 $-C^*(N_1) \coprod -C^*(N_2)$ is preserved by base change as resolution of $N_1 \coprod N_2$. It follows that the resolution $-C^*(N_i)$ is preserved by base change for $i = 1, 2$. \square

Corollary 4.7. *Assume BC1-2 and AB1-3 as in Proposition 4.6. If N is in $\hat{X}^{\text{fl}}(T)$ then any DX-resolution $-C^*(N) \twoheadrightarrow N$ and any $\hat{D}D$ -coresolution $N \twoheadrightarrow +C^*(N)$ as in Section 2.2 is preserved by base change.*

Proof. By Theorem 4.4 there exist a DX-resolution and a $\hat{D}D$ -coresolution in \hat{X}^{fl} . The result follows from Proposition 4.6 (iii) and Lemma 4.3. \square

Definition 4.8. Let A be a category fibred in abelian categories over C and let $D \subseteq A$ be an inclusion morphism of a category fibred in additive categories. For T in C let $\hat{D}^{\text{fl}}(T)$ denote the full subcategory of $A(T)$ of objects K with a finite coresolution $K \rightarrow L^*$ with objects L^i in $\hat{D}^{\text{fl}}(T)$ for $i \geq 0$.

Lemma 4.9. *With these notions we have:*

- (i) *Epimorphisms in $\hat{D}^{\text{fl}}(T)$ are admissible if and only if $\hat{D}^{\text{fl}}(T) = \check{D}^{\text{fl}}(T)$.*

Assume BC1-2 and AB1-4 for $(A(T), X(T), D(T))$. Then:

- (ii) $\hat{D}^{\text{fl}}(T) = \hat{D}(T) \cap \check{D}^{\text{fl}}(T)$.
- (iii) *Epimorphisms in $\hat{D}^{\text{fl}}(T)$ are admissible if epimorphisms in $\hat{D}(T)$ are admissible.*

Proof. (i) is trivially true. In (ii) $\hat{D}^{\text{fl}}(T) \subseteq \hat{D}(T) \cap \check{D}^{\text{fl}}(T)$ is obvious. For the other inclusion, suppose K is an object in $\hat{D}(T) \cap \check{D}^{\text{fl}}(T) \setminus \hat{D}^{\text{fl}}(T)$ with a $\hat{D}^{\text{fl}}(T)$ -coresolution $K \rightarrow L^*$ of length $n > 0$. Since monomorphisms are admissible in $\hat{D}(T) = \hat{X}(T) \cap X(T)^\perp$, all $K^i = \ker(L^i \rightarrow L^{i+1})$ are contained in $\hat{D}(T)$, and we can assume $n = 1$. But then K has to be in $\hat{X}^{\text{fl}} \cap \hat{D} = \hat{D}^{\text{fl}}(T)$ by Proposition 4.6. Since (i) is true “without the fl” by [7, 4.1], (iii) follows immediately from (i) and (ii). \square

5. COHEN-MACAULAY APPROXIMATION OF FLAT FAMILIES

We define fibred categories of Cohen-Macaulay maps with flat modules and show that they allow Cohen-Macaulay approximation in the finite type case and the local, algebraic case.

5.1. The finite type case. Let $h : S \rightarrow T$ be a ring homomorphism of noetherian rings. We say that h is a *Cohen-Macaulay (CM) map* if it is of finite type, faithfully flat and all fibres are Cohen-Macaulay (cf. [19, 6.8.1]). In particular h is equidimensional ([20, 15.4.1]). B. Conrad has defined the *dualising module* ω_h for any CM h , see [12, Sec. 3.5]. Suppose h has pure relative dimension n . For some $N \geq n$ there is a surjective S -algebra map $P \rightarrow T$ where $P = S[t_1, \dots, t_N]$. Let $\omega_{P/S} := \wedge^N \Omega_{P/S}$. Then there is an isomorphism $\omega_h \cong \text{Ext}_P^{N-n}(T, \omega_{P/S})$ which is natural in the factorisation $S \rightarrow P \rightarrow T$, see [12, 3.5.3-6]. By (local) duality theory and Corollary 2.9 ω_h is S -flat (or see [12, Cor. 3.5.2]). If S is a field, we have that ω_h is a canonical module of T as in Example 2.1, cf. [10, 3.3.7 and 16].

Let CM be the category with objects the CM maps and morphisms $(g, f) : h_1 \rightarrow h_2$ pairs of ring homomorphisms $g : S_1 \rightarrow S_2$ and $f : T_1 \rightarrow T_2$ such that $h_2 g = f h_1$

and such that the induced map $f \otimes 1 : T_1 \otimes S_2 \rightarrow T_2$ is an isomorphism:

$$\begin{array}{ccc} T_1 & \xrightarrow{f} & T_2 \xleftarrow{\cong} T_1 \otimes_{S_1} S_2 \\ h_1 \uparrow & & \uparrow h_2 \\ S_1 & \xrightarrow{g} & S_2 \end{array}$$

Let \mathbf{NR} denote the category of noetherian rings. The forgetful functor $p : \mathbf{CM} \rightarrow \mathbf{NR}$; $(g, f) \mapsto g$, makes \mathbf{CM} fibred in groupoids over \mathbf{NR} . The essential part is that \mathbf{CM} should allow base change, i.e. given $g : S_1 \rightarrow S_2$ and $h_1 : S_1 \rightarrow T_1$ as above there should exist a T_2 , an $h_2 : S_2 \rightarrow T_2$ and an f such that (g, f) is a morphism $h_1 \rightarrow h_2$ in \mathbf{CM} . This follows from [20, 15.4.3].

Let \mathbf{mod} be the category of pairs $(h : S \rightarrow T, N)$ with h in \mathbf{CM} and N a finite T -module. A morphism $(h_1, N_1) \rightarrow (h_2, N_2)$ is a morphism $(g, f) : h_1 \rightarrow h_2$ in \mathbf{CM} and a f -linear map $\alpha : N_1 \rightarrow N_2$. Then α is cocartesian with respect to the forgetful functor $F : \mathbf{mod} \rightarrow \mathbf{CM}$ if $1 \otimes \alpha : T_2 \otimes N_1 \rightarrow N_2$ is an isomorphism. It follows that \mathbf{mod} is fibred in abelian categories over \mathbf{CM} . Adding the property that N is S -flat gives the full subcategory \mathbf{mod}^{fl} . Moreover, let \mathbf{MCM} be the full subcategory of \mathbf{mod}^{fl} where the fibre $N_s = N \otimes_S k(s)$ is a maximal Cohen-Macaulay T_s -module for all $s \in \text{Spec } S$. The inclusions $\mathbf{MCM} \subseteq \mathbf{mod}^{\text{fl}} \subseteq \mathbf{mod}$ are inclusion morphisms of categories fibred in additive categories (FAdS) over \mathbf{CM} . For \mathbf{MCM} this follows from [20, 15.4.3]. If h is a \mathbf{CM} map let $\mathbf{mod}_h, \mathbf{MCM}_h, \dots$ denote the fibre categories of $\mathbf{mod}, \mathbf{MCM}, \dots$ over h . An object in \mathbf{MCM}_h is called an (h) -family of maximal Cohen-Macaulay modules.

Given a morphism $h_1 \rightarrow h_2$ in \mathbf{CM} . By [12, Thm. 3.6.1] there is a natural isomorphism with base change $T_2 \otimes \omega_{h_1} \cong \omega_{h_2}$ which is compatible with localisation of T_1 and is functorial with respect to composition $h_1 \rightarrow h_2 \rightarrow h_3$. It follows that $h \mapsto (h, \omega_h)$ defines a morphism $\omega : \mathbf{CM} \rightarrow \mathbf{MCM}$ of fibred categories over \mathbf{NR} which is a section of the forgetful $F : \mathbf{MCM} \rightarrow \mathbf{CM}$. Let \mathbf{D} be the full subcategory of \mathbf{MCM} over \mathbf{CM} with the objects (h, D) where D is an object in $\text{Add}\{\omega_h\}$. The inclusion $\mathbf{D} \subseteq \mathbf{MCM}$ is an inclusion of FAdS over \mathbf{CM} .

If \mathbf{U} denotes any of these FAdS over \mathbf{CM} , let $\underline{\mathbf{U}}$ denote the quotient ('stable') category \mathbf{U}/\mathbf{D} . With this notation we have the following.

Theorem 5.1. *The pair $(\mathbf{mod}, \mathbf{MCM})$ over \mathbf{CM} satisfies BC1-2 and the triple of fibre categories $(\mathbf{mod}_h, \mathbf{MCM}_h, \mathbf{D}_h)$ satisfies AB1-4 for all objects h in \mathbf{CM} . Moreover:*

- (i) *The fibred categories $\widehat{\mathbf{MCM}}^{\text{fl}}$ and $\hat{\mathbf{D}}^{\text{fl}}$ equals \mathbf{mod}^{fl} and $\hat{\mathbf{D}} \cap \mathbf{mod}^{\text{fl}}$ respectively.*
- (ii) *For any object (h, N) in \mathbf{mod}^{fl} , N admits an \mathbf{MCM} -approximation and a $\hat{\mathbf{D}}^{\text{fl}}$ -hull which in particular are preserved by any base change.*
- (iii) *The \mathbf{MCM} -approximation induces a morphism of categories fibred in additive categories $j^! : \underline{\mathbf{mod}}^{\text{fl}} \rightarrow \underline{\mathbf{MCM}}$ which is a right adjoint to the full and faithful inclusion morphism $j_! : \underline{\mathbf{MCM}} \rightarrow \underline{\mathbf{mod}}^{\text{fl}}$.*
- (iv) *The $\hat{\mathbf{D}}^{\text{fl}}$ -hull induces a morphism of categories fibred in additive categories $i^* : \underline{\mathbf{mod}}^{\text{fl}} \rightarrow \underline{\hat{\mathbf{D}}^{\text{fl}}}$ which is a left adjoint to the full and faithful inclusion morphism $i_* : \underline{\hat{\mathbf{D}}^{\text{fl}}} \rightarrow \underline{\mathbf{mod}}^{\text{fl}}$.*
- (v) *Together these maps give the following commutative diagram of short exact sequences of categories fibred in additive categories:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \hat{\mathbf{D}}^{\text{fl}} & \xrightarrow{i_*} & \underline{\mathbf{mod}}^{\text{fl}} & \xrightarrow{j^!} & \underline{\mathbf{MCM}} \longrightarrow 0 \\ & & \text{id} \downarrow & & \parallel & & \uparrow \text{id} \\ 0 & \longleftarrow & \underline{\hat{\mathbf{D}}^{\text{fl}}} & \xleftarrow{i^*} & \underline{\mathbf{mod}}^{\text{fl}} & \xleftarrow{j_!} & \underline{\mathbf{MCM}} \longleftarrow 0 \end{array}$$

Proof. Since base change is given by the tensor product BC1 and BC2 follows. In particular, a short exact sequence $e : N_1 \rightarrow N_2 \rightarrow N_3$ in mod_h with N_3 and either N_1 or N_2 in MCM_h gives short exact sequences of MCMs after base change to each fibre T_s , i.e. AB1 and AB4.

For AB2, suppose M is in MCM_h . Then $M^\vee = \text{Hom}_T(M, \omega)$ is in MCM_h too by Corollary 2.9. Since M^\vee is finite there is a short exact sequence $M^\vee \leftarrow T^r \leftarrow M_1$. By AB1 M_1 is in MCM_h . Applying $\text{Hom}_T(-, \omega_h)$ gives the desired short exact sequence since Corollary 2.9 implies that $\text{Ext}_T^1(M^\vee, \omega_h) = 0$ and that the natural map $M \rightarrow M^{\vee\vee}$ is an isomorphism. AB3 also follows from Corollary 2.9.

Any N in $\widehat{\text{MCM}}_h^{\text{fl}}$ has by definition a finite MCM-resolution (say of length n) preserved by base change. Since objects in MCM are S -flat it follows by induction on n that $\text{Tor}_1^S(N, k(s)) = 0$ for all $s \in \text{Spec } S$. Hence N is S -flat. Conversely, if N is in mod_h^{fl} it follows that a sufficiently high syzygy of N is in MCM_h , i.e. N is in $\widehat{\text{MCM}}_h^{\text{fl}}$. With Proposition 4.6 this gives $\hat{D}_h^{\text{fl}} = \text{mod}_h^{\text{fl}} \cap \text{MCM}_h^\perp$. By induction on the length of the resolution $\hat{D}_h \subseteq \text{MCM}_h^\perp$ and so $\text{mod}_h^{\text{fl}} \cap \hat{D}_h \subseteq \hat{D}_h^{\text{fl}}$. The opposite inclusion is clear by the first part of (i). Now (ii)-(v) follows directly from Theorem 4.5. \square

Corollary 5.2. *Let $h : S \rightarrow T$ be an object in CM. Then:*

- (i) $D_h = \text{MCM}_h^\perp \cap \text{MCM}_h$ and $\hat{D}_h^{\text{fl}} = \text{MCM}_h^\perp \cap \text{mod}_h^{\text{fl}}$
- (ii) *The kernel of a surjective map in \hat{D}_h^{fl} is contained in \hat{D}_h^{fl} .*

Proof. (ii): Note that if $N_1 \rightarrow N_2 \rightarrow N_3$ is a short exact sequence with N_2 and N_3 in \hat{D}_h^{fl} , in particular S -flat, then N_1 has to be S -flat too. To show that N_1 is in \hat{D}_h we use the criterion in [7, 4.6] to show that $\check{D}_h = \hat{D}_h$: Suppose that M is in MCM_h . Assume that M satisfies $\text{MCM}_h\text{-inj.dim } M = n < \infty$ which by [7, 4.3] is equivalent to the existence of a coresolution of M of length n in D_h . The fibre at any $s \in \text{Spec } S$ gives a D_{T_s} -coresolution of the MCM T_s -module M_s . Since $\check{D}_{T_s} = \hat{D}_{T_s}$ ([7, 6.3]) it follows by [7, 4.6] that M_s is contained in D_{T_s} . Since $D_{T_s} = \text{MCM}_{T_s} \cap \text{MCM}_{T_s}^\perp$ by [7, 3.7] it follows from Corollary 2.9 that $\text{MCM}_h\text{-inj.dim } M = 0$ and so M is in $\text{MCM}_h \cap \text{MCM}_h^\perp$. But by Theorem 5.1 we can invoke [7, 3.7] again which gives $\text{MCM}_h \cap \text{MCM}_h^\perp = D_h$ so M is in D_h . By [7, 4.6(d)] $\check{D}_h = \hat{D}_h$ follows and N_1 is in \hat{D}_h . \square

Remark 5.3. Let Λ be any noetherian ring. By abuse of notation let $\text{CM} \rightarrow {}_\Lambda \mathbf{R}$ denote the category fibred in groupoids obtained by restriction to the category of noetherian Λ -algebras ${}_\Lambda \mathbf{R}$. Fix a Cohen-Macaulay map $\Lambda \rightarrow T^\circ$. There is a section $s : {}_\Lambda \mathbf{R} \rightarrow \text{CM}$ defined by $s : S \mapsto (S \rightarrow T^\circ \otimes S)$. Let \mathbf{T}° denote the resulting fibred subcategory of CM. We restrict mod , MCM and D to \mathbf{T}° and obtain categories fibred in abelian and additive categories over \mathbf{T}° respectively. These restricted fibred categories satisfy the axioms AB1-4 and BC1-2 and we obtain restricted versions of Theorem 5.1 and Corollary 5.2.

Let \mathbf{P} denote the fibred subcategory of MCM over CM of pairs (h, P) with $h : S \rightarrow T$ in CM and P a finite projective T -module. Let $\hat{\mathbf{P}}^{\text{fl}}$ denote the full subcategory of mod^{fl} of pairs (h, Q) such that Q has a finite projective dimension. The inclusions of categories fibred in additive categories $\mathbf{P} \subseteq \hat{\mathbf{P}}^{\text{fl}} \subseteq \text{mod}^{\text{fl}}$ are closed under extensions over CM.

Lemma 5.4. *There is an exact equivalence $\hat{D}^{\text{fl}} \simeq \hat{\mathbf{P}}^{\text{fl}}$ of categories fibred in additive categories defined by the functor $(h, L) \mapsto (h, \text{Hom}_T(\omega_h, L))$ with a quasi-inverse $(h, Q) \mapsto (h, Q \otimes_T \omega_h)$. It induces an equivalence of fibred quotient categories $\hat{D}^{\text{fl}}/D \simeq \hat{\mathbf{P}}^{\text{fl}}/\mathbf{P}$.*

Proof. By base change we reduce to the absolute case where the equivalence is well known, cf. [24, I 4.10.16]. The functor $\text{Hom}_-(\omega_-, -)$ takes a map $\varphi : L_1 \rightarrow L_2$ over $h_1 \rightarrow h_2$ to the map $\text{Hom}_{T_1}(\omega_{h_1}, L_1) \rightarrow \text{Hom}_{T_2}(\omega_{h_2}, L_2)$; $\alpha \mapsto (\varphi\alpha)^\#$, the unique map which pulls back to $\varphi\alpha$ by the cocartesian map $\omega_{h_1} \rightarrow \omega_{h_2}$. It is a well defined functor since the dualising module is functorial. If L is in \hat{D}_h^{fl} then $\text{Ext}_{T_s}^i(\omega_{T_s}, L_s) = 0$ for all $i \neq 0$ and $s \in \text{Spec } S$. By Corollary 2.9 the functor is exact and $\text{Hom}_T(\omega_h, L)$ is S -flat. In particular $\text{End}_T(\omega_h) \cong T$. If D is in D_h then $\text{Hom}_T(\omega_h, D)$ is projective as a direct summand of a free module. If $D^* \rightarrow L$ is a finite D_h -resolution of L , then $\text{Hom}_T(\omega_h, D^*)$ gives a projective resolution of $\text{Hom}_T(\omega_h, L)$ since $\hat{D}_h^{\text{fl}} \subseteq \text{MCM}_h^\perp$ (Corollary 5.2). The natural map $\text{ev} : \text{Hom}_T(\omega_h, L) \otimes \omega_h \rightarrow L$ commutes with base change to the fibres where it is an isomorphism ([10, 9.6.5]) and Nakayama's lemma and S -flatness implies that ev is an isomorphism too. Let $P^* \rightarrow Q$ be a P -resolution of Q in \hat{P}_h^{fl} of length n . Define covariant functors $G^i : \text{mod}_S \rightarrow \text{mod}_T$ by $G^i(V) := H^{i-n}(P^* \otimes_T \omega_h \otimes_S V)$. Since $P^* \otimes_T \omega_h$ is S -flat, $\{G^i\}$ defines a cohomological δ -functor. Since $P^* \otimes k(s)$ gives a resolution of $Q \otimes_S k(s)$ for all $s \in \text{Spec } S$, [10, 9.6.5] and Proposition 2.5 implies that $G^n(S) = Q \otimes_T \omega_h$ is S -flat and $P^* \otimes_T \omega_h \rightarrow Q \otimes_T \omega_h$ is a D -resolution. Moreover, the natural map $Q \rightarrow \text{Hom}_T(\omega_h, Q \otimes \omega_h)$ is an isomorphism by Nakayama's lemma again. \square

Example 5.5. Assume A is a Cohen-Macaulay ring with a canonical module ω_A . Given $L_i \in \hat{D}_A$ and put $Q_i = \text{Hom}_A(\omega_A, L_i)$ for $i = 1, 2$. If I is an injective resolution of L_2 and P is a projective resolution of Q_1 then both spectral sequences of $\text{Hom}_A(P, \text{Hom}_A(\omega_A, I))$ collapse at page 2 (use [24, I 4.10.19]) to give canonical isomorphisms

$$(5.5.1) \quad \text{Ext}_A^*(L_1, L_2) \cong \text{Ext}_A^*(Q_1, Q_2)$$

Remark 5.6. In his unpublished manuscript Buchweitz gave a construction of Cohen-Macaulay approximation for finite A -modules if A is a not necessarily commutative Gorenstein ring, see [11]. The MCM -approximation and the \hat{D}^{fl} -hull in Theorem 5.1 can be given essentially by the same construction. Let N be a finite T -module which is S -flat where $h : S \rightarrow T$ is a finite type Cohen-Macaulay map. Let $P = P(N) \rightarrow N$ be a projective resolution of N (i.e. by finite projective T -modules). Then $P^\vee = \text{Hom}_T(P, \omega_h)$ is a bounded below complex with bounded cohomology because $\text{Ext}_T^i(N, \omega_h) = 0$ for i greater than the relative dimension d of h by Corollary 2.9 ($\text{inj.dim } \omega_{T_s} = \dim T_s \leq d$). Then we can choose a projective resolution $f : P(P^\vee) \rightarrow P^\vee$ of P^\vee which is bounded above. Let $C = C(f)$ be the mapping cone of f . The modules in C are direct sums of projective modules and modules in D_h and the (co)kernels in the acyclic C are modules in MCM_h . By Corollary 2.9 it follows that C^\vee is acyclic too. There is a composition of natural maps $P \cong P^{\vee\vee} \rightarrow P(P^\vee)^\vee := G$ which hence is a quasi-isomorphism. But then G is just the representing complex of N and the MCM -approximation and the \hat{D} -hull is obtained as in Section 2.2. In the the case of coherent rings with a cotilting module (a concept introduced by Y. Miyashita, see [38, p. 142]) J.-i. Miyachi implicitly gave the same construction in [37, 3.2].

Suppose h has pure relative dimension and N_s is a non-zero Cohen-Macaulay T_s -module with $n = \dim T_s - \dim N_s$ constant for all $s \in \text{Spec } S$, i.e. N is a family of Cohen-Macaulay modules of codepth n . Put $N^\vee := \text{Ext}_T^n(N, \omega_h)$. Then $P^\vee = P(N)^\vee$ is quasi-isomorphic to $H^n(P^\vee) = N^\vee$ by duality theory and Corollary 2.9. So if $(F, d) \rightarrow N^\vee$ is a projective resolution of N^\vee , the representing complex is given as $G = F^\vee$. If $\text{Syz}_i^T N^\vee$ denotes the i^{th} syzygy module $\text{im } d_i$ and d_i^\vee denotes $\text{Hom}_T(d_i, \omega_h)$ then the commutative diagram (2.0.9) with short exact sequences is

given as

$$\begin{array}{ccccc}
 \mathrm{im}(d_n^\vee) & \longrightarrow & \mathrm{Hom}_T(\mathrm{Syz}_n^T N^\vee, \omega_h) & \longrightarrow & N^{\vee\vee} \\
 \parallel & & \downarrow & \square & \downarrow \\
 \mathrm{im}(d_n^\vee) & \longrightarrow & \mathrm{Hom}_T(F_n, \omega_h) & \longrightarrow & \mathrm{coker}(d_n^\vee) \\
 & & \downarrow & & \downarrow \\
 & & \mathrm{Hom}_T(\mathrm{Syz}_{n+1}^T N^\vee, \omega_h) & = & \mathrm{Hom}_T(\mathrm{Syz}_{n+1}^T N^\vee, \omega_h)
 \end{array}$$

with the MCM_h -approximation and \hat{D}_h -hull of N given by the upper horizontal and right vertical sequence, respectively, since $N^{\vee\vee} \cong N$ by duality theory and Corollary 2.9. Let C denote the mapping cone of a comparison map $P(N) \rightarrow G$. The homology of the truncated short exact sequence of complexes $G \rightarrow C \rightarrow P(N)[-1]$ gives the MCM_h -approximation $\mathrm{coker} d_{i+2}^G \rightarrow \mathrm{coker} d_{i+2}^C \rightarrow \mathrm{Syz}_i^T N$ for $i \geq 0$ (with $i = 0$ being the upper horizontal sequence in the diagram). In the absolute case (with N Cohen-Macaulay) this latter construction of the MCM approximation of $\mathrm{Syz}_i N$ was given by J. Herzog and A. Martsinkovsky in [27, 1.1].

5.2. Local cases.

We formulate local variants of the approximation theorem.

Fix a field k . Let \mathbf{H} denote the category of noetherian, henselian, local rings S with residue field $S/\mathfrak{m}_S \cong k$ and with local ring homomorphisms. A map $h : S \rightarrow T$ in \mathbf{H} is *algebraic* (or T is an algebraic S -algebra) if there is a finite type S -algebra \tilde{T} and a maximal ideal \mathfrak{m} in \tilde{T} with $\tilde{T}/\mathfrak{m} \cong k$ such that h is given by henselisation of \tilde{T} in \mathfrak{m} . Fix an algebraic k -algebra A which is supposed to be Cohen-Macaulay. Objects in the category \mathbf{hCM} are algebraic and flat S -algebras T with $T \otimes_S k \cong A$. A morphism $h_1 \rightarrow h_2$ is a pair of commuting maps $f : T_1 \rightarrow T_2$ and $g : S_1 \rightarrow S_2$ in \mathbf{H} as for the finite type case, giving a cocartesian square. Base change exists for the forgetful functor $\mathbf{hCM} \rightarrow \mathbf{H}$ and is given by the henselisation of the tensor product $T = T_1 \otimes_{S_1} S_2$ in the maximal ideal $\mathfrak{m}_{T_1} T + \mathfrak{m}_{S_2} T$. We denote it by $T_1 \tilde{\otimes}_{S_1} S_2$. It follows that $\mathbf{hCM} \rightarrow \mathbf{H}$ is fibred in groupoids. The objects in \mathbf{hCM} will be called henselian Cohen-Macaulay (hCM) maps.

If $\tilde{h} : \tilde{S} \rightarrow \tilde{T}$ is a finite type CM map and t a k -point in $\mathrm{Spec} \tilde{T}$ with image s in $\mathrm{Spec} \tilde{S}$ then localisation for the étale topology at t and s gives a hCM map $h : S = \tilde{S}^h \rightarrow \tilde{T}^h = T$. Conversely, every hCM map is obtained this way which follows from [21, 18.6.6 and 18.6.10] and [20, 15.4.3 and 12.1.1]. We will call such an \tilde{h} a (finite type) representative of h . The dualising module $\omega_{\tilde{h}}$ induces an S -flat finite T -module ω_h called the dualising module for h . Two representatives of h factor through a common étale neighbourhood contained in CM and since the dualising module is functorial for CM the dualising module ω_h is functorial too.

Let \mathbf{mod} denote the category of pairs $(h : S \rightarrow T, \mathcal{N})$ with h in \mathbf{hCM} and \mathcal{N} a finite T -module. Morphisms are defined as for the finite type case and the forgetful functor $\mathbf{mod} \rightarrow \mathbf{hCM}$ makes \mathbf{mod} fibred in abelian categories over \mathbf{hCM} . Let $\mathbf{mod}^{\mathrm{fl}}$ denote the full subcategory of objects (h, \mathcal{N}) in \mathbf{mod} with \mathcal{N} S -flat and let \mathbf{MCM} denote the full subcategory of objects (h, \mathcal{N}) in $\mathbf{mod}^{\mathrm{fl}}$ where the closed fibre $\mathcal{N} \otimes_S k$ is a maximal Cohen-Macaulay A -module. Let \mathbf{D} denote the full subcategory of \mathbf{MCM} of objects (h, D) with D in $\mathrm{Add}\{\omega_h\}$. All three are FAd subcategories of \mathbf{mod} over \mathbf{hCM} . Any finite T -module \mathcal{N} has finite presentation hence it is induced from a finite module over a representative of T . If \mathcal{N} is contained in one of the subcategories the representative can be assumed to belong to the corresponding finite type subcategory (*loc. sit.*). Similarly all maps in these fibred categories over \mathbf{H} are induced from maps in the corresponding fibred categories of finite type objects.

Let $\mathbf{L}(\mathbf{C})$ denote the category of noetherian, (complete) local rings with residue field k and local ring homomorphisms. Let \mathbf{ICM} (\mathbf{cCM}) denote the category of local Cohen-Macaulay (\mathbf{ICM}) maps (respectively complete Cohen-Macaulay or \mathbf{cCM} maps) defined analogously as above with (completion of) *essentially of finite type* replacing algebraic. Similar arguing as above makes \mathbf{ICM} (\mathbf{cCM}) fibred in groupoids over $\mathbf{L}(\mathbf{C})$. The definitions of the module categories apply in the local and the complete case too and we use the same notation in all three cases. Again objects and maps are induced from the finite type case.

Either arguing with representatives or applying the proofs for Theorem 5.1 and Corollary 5.2 (with only minor adjustments) we obtain the following.

Corollary 5.7. *Let \mathbf{xCM} denote either \mathbf{hCM} , \mathbf{ICM} or \mathbf{cCM} . The pair $(\mathbf{mod}, \mathbf{MCM})$ of fibred categories over \mathbf{xCM} satisfies BC1-2 and the triple of fibre categories $(\mathbf{mod}_h, \mathbf{MCM}_h, \mathbf{D}_h)$ satisfies AB1-4 for all objects h in \mathbf{xCM} .*

Moreover; the statements (i-v) in Theorem 5.1 and (i-ii) in Corollary 5.2 are valid over \mathbf{xCM} too.

6. MINIMAL APPROXIMATIONS AND SEMI-CONTINUITY OF INVARIANTS

We show that the Cohen-Macaulay approximation and the $\hat{\mathbf{D}}$ -hull in Corollary 5.7 can be chosen to be minimal. Upper semi-continuous invariants on \mathbf{MCM}_A or \mathbf{FID}_A extends to upper semi-continuous invariants on \mathbf{mod}_A . An example is given by the ω_A -ranks in the representing \mathbf{D} -complex.

Lemma 6.1. *Let $S \rightarrow T$ be a homomorphism of noetherian rings and \mathfrak{a} an ideal in S such that $I = \mathfrak{a}T$ is contained in the Jacobson radical of T . Let M and N be finite T -modules. Let $T_n = T/I^{n+1}$, $M_n = T_n \otimes M$ and $N_n = T_n \otimes N$. Suppose there exists a tower of surjections $\{\varphi_n : M_n \rightarrow N_n\}$. Fix any non-negative integer n_0 . Then there exists a T -linear surjection $\psi : M \rightarrow N$ such that $T_{n_0} \otimes \psi = \varphi_{n_0}$. If the φ_n are isomorphisms and N is S -flat then ψ is an isomorphism.*

Proof. Let $\hat{T} = \varprojlim T_n$, $\hat{M} = \varprojlim M_n$ and $\hat{N} = \varprojlim N_n$. We have

$$(6.1.1) \quad \varprojlim \mathrm{Hom}_{T_n}(M_n, N_n) \cong \mathrm{Hom}_{\hat{T}}(\hat{M}, \hat{N}) \cong \hat{T} \otimes \mathrm{Hom}_T(M, N).$$

Hence $\varprojlim \varphi_n = \Sigma r^{(i)} \otimes \beta^{(i)}$ with $r^{(i)} \in \hat{T}$ and $\beta^{(i)} \in \mathrm{Hom}_T(M, N)$. Let $r_{n_0}^{(i)}$ be the image of $r^{(i)}$ under $\hat{T} \rightarrow T_{n_0}$ and choose liftings $t^{(i)}$ in T of $r_{n_0}^{(i)}$. Put $\psi = \Sigma t^{(i)} \beta^{(i)}$. Then $T_{n_0} \otimes \psi = \varphi_{n_0}$. Since $T_{n_0} \otimes \mathrm{coker} \psi = \mathrm{coker} \varphi_{n_0} = 0$, Nakayama's lemma implies $\mathrm{coker} \psi = 0$. Since $T_{n_0} \otimes_T N$ equals $N \otimes_S S/\mathfrak{a}^{n_0+1}$, S -flatness of N implies $T_{n_0} \otimes \ker \psi = \ker \varphi_{n_0}$ and if $\ker \varphi_{n_0} = 0$ then Nakayama's lemma implies $\ker \psi = 0$. \square

Proposition 6.2. *Let \mathbf{xCM} denote either \mathbf{hCM} , \mathbf{ICM} or \mathbf{cCM} , let $h : S \rightarrow T$ be an object in \mathbf{xCM} and let $\xi : \mathcal{L} \rightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{N}$ be an \mathbf{MCM}_h -approximation of \mathcal{N} . Then the following statements are equivalent.*

- (i) *The sequence ξ is a right minimal \mathbf{MCM}_h -approximation.*
- (ii) *There are no surjections $\mathcal{M} \rightarrow \omega_h$ which induces a surjection $\mathcal{L} \rightarrow \omega_h$.*
- (iii) *There are no common ω_h -summand in $\mathcal{L} \rightarrow \mathcal{M}$.*
- (iv) *The closed fibre $\xi \otimes_{Sk}$ is a right minimal \mathbf{MCM}_A -approximation.*
- (v) *The completion $(\xi \otimes_{Sk})^\wedge$ of the closed fibre is a right minimal $\mathbf{MCM}_{\hat{A}}$ -approximation.*

The analogous statements (i')-(v') for a $\hat{\mathbf{D}}_h^{\mathfrak{a}}$ -hull $\xi' : \mathcal{N}' \rightarrow \mathcal{L}' \rightarrow \mathcal{M}'$ are equivalent. In particular:

- (i') *The sequence ξ' is a left minimal $\hat{\mathbf{D}}_h^{\mathfrak{a}}$ -hull.*
- (ii') *There are no surjections $\mathcal{M}' \rightarrow \omega_h$ which induces a surjection $\mathcal{L}' \rightarrow \omega_h$.*

Proof. Suppose there is a surjection $\mathcal{M} \rightarrow \omega_h$ such that the composition $\mathcal{L} \rightarrow \omega_h$ is surjective too. Then the kernels of these maps give a new MCM-approximation of \mathcal{N} by Corollary 5.7. Since a surjection $\mathcal{L} \rightarrow \omega_h$ has to split by Corollary 5.7, ω_h is a common summand in $\mathcal{L} \rightarrow \mathcal{M}$ and π cannot be right minimal.

Let the closed fibre $\xi_{\otimes_S k}$ of the sequence ξ be denoted by $L \rightarrow M \xrightarrow{p} N$. Assume there is a non-surjective endomorphism $\theta : \mathcal{M} \rightarrow \mathcal{M}$ with $\pi\theta = \pi$. Then $\theta_0 = \theta_{\otimes_S k}$ gives a non-surjective endomorphism of M with $p\theta_0 = p$. It follows that the completion $\hat{L} \rightarrow \hat{M} \rightarrow \hat{N}$ is not a right minimal Cohen-Macaulay approximation of \hat{N} . By [24, 1.12.8] there is a common $\omega_{\hat{A}}$ -summand in $\hat{L} \rightarrow \hat{M}$. Let $\varphi : \hat{M} \rightarrow \omega_{\hat{A}}$ denote the projection. By Lemma 6.1 there exists a surjection $\psi : M \rightarrow \omega_A$. The induced map $L \rightarrow \omega_A$ is surjective too. The map ψ lifts to a surjection $\mathcal{M} \rightarrow \omega_h$ (with $\mathcal{L} \rightarrow \omega_h$ surjective) since the canonical map $\text{Hom}_T(\mathcal{M}, \omega_h) \rightarrow \text{Hom}_A(M, \omega_A)$ is surjective by Corollary 2.9. The \hat{D}^{fl} -case is analogous. \square

Corollary 6.3. *Let xCM denote either hCM, ICM or cCM. For any object (h, \mathcal{N}) in the fibred category mod^{fl} over xCM , \mathcal{N} admits a right minimal MCM-approximation and a left minimal \hat{D}^{fl} -hull which remain minimal after base change and which in particular are unique up to non-canonical isomorphism.*

Proof. The existence of a right minimal ξ follows immediately from criterion (iii) in Proposition 6.2. Moreover ξ is right minimal if and only if the closed fibre $\xi_{\otimes_S k}$ is right minimal. Since any base change ξ_1 of ξ has the same closed fibre as ξ , ξ_1 is right minimal if ξ is. \square

Remark 6.4. In [25, 2.3] M. Hashimoto and A. Shida gave essentially the analog of Proposition 6.2 in the absolute case of a Cohen-Macaulay Zariski local ring with a canonical module. They attributed the complete case to Y. Yoshino. Note Miyachi's proof in the cotilting semi-perfect case, see [37, 3.4] (cf. [24, 1.12.8]). In [43, 3.1] A.-M. Simon and J. R. Strooker give a short independent proof. The proof of Proposition 6.2 also works in the Zariski local case in full generality and is different from these (but depends on the complete case).

Since minimal choices of MCM_A -approximations and \hat{D}_A -hulls exist and are unique up to isomorphism any invariant defined for MCM modules or for FID modules is extended to all finite A -modules. Upper semi-continuity of the invariants is also extended as we explain now. First some notation.

Let $h : S \rightarrow T$ be ring homomorphism and \mathcal{N} a finite T -module. If $t \in \text{Spec } T$ has image $s \in \text{Spec } S$, let $\mathcal{N}_{\mathfrak{p}_t}$ denote the localisation of \mathcal{N} at the prime ideal t , let $h_{\mathfrak{p}_t} : S_{\mathfrak{p}_s} \rightarrow T_{\mathfrak{p}_t}$ denote the ring homomorphism obtained by localising, and put $\mathcal{N}(t) = \mathcal{N}_{\mathfrak{p}_t} \otimes_{S_{\mathfrak{p}_s}} k(s)$ which is a $T(t)$ -module; indeed $\mathcal{N}(t) \cong T(t) \otimes_{T_{\mathfrak{p}_t}} \mathcal{N}_{\mathfrak{p}_t}$. If h is a finite type Cohen-Macaulay map, $h_{\mathfrak{p}_t}$ is in ICM.

Suppose μ is an invariant on MCM_A where A is a Cohen-Macaulay local ring with canonical module. Let $_{\text{MCM}}\mu$ denote the induced invariant on mod_A defined by $_{\text{MCM}}\mu(N) = \mu(M)$ where $L \rightarrow M \rightarrow N$ is the minimal Cohen-Macaulay approximation of N . Similarly $_{\text{FID}}\mu(N) = \mu(L)$ for an invariant μ defined on FID_A . Use the minimal hull $N \rightarrow L' \rightarrow M'$ to define $_{\text{FID}}\mu'$ and $_{\text{MCM}}\mu'$.

The following theorem is a major application of what we have done so far.

Theorem 6.5. *Let μ be an additive non-negative numerical invariant defined for maximal Cohen-Macaulay modules or for finite modules of finite injective dimension on a Cohen-Macaulay local ring with canonical module. Assume μ is upper semi-continuous for finite type flat families $(h : S \rightarrow T, \mathcal{M})$ in MCM (or in \hat{D}). Then the induced invariants $_{\text{MCM}}\mu$ and $_{\text{MCM}}\mu'$ (or $_{\text{FID}}\mu$ and $_{\text{FID}}\mu'$) are upper semi-continuous in finite type flat families (h, \mathcal{N}) in mod^{fl} .*

Proof. Given $(h : S \rightarrow T, \mathcal{N})$ in $\mathbf{mod}^{\mathfrak{fl}}$ and $t \in \mathrm{Spec} T$. By Corollary 6.3 there exists open affines $U = \mathrm{Spec} S_1 \subseteq \mathrm{Spec} S$, $V = \mathrm{Spec} T_1 \subseteq \mathrm{Spec} T$ with $t \in V$, $h_1 : S_1 \rightarrow T_1$ induced from h , and a MCM_{h_1} -approximation $\xi : \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N}|_V$ such that the localisation $\xi_{\mathfrak{p}_t}$ is minimal. By Corollary 6.3 $\xi(t)$ is minimal too. Put $n = \mu(\mathcal{M}(t))$. Since μ is upper semi-continuous there is an open $V_n \subseteq V$ containing t such that $\mu(\mathcal{M}(t')) \leq n$ for all $t' \in V_n$. If $L \rightarrow M \rightarrow \mathcal{N}(t')$ is the minimal MCM approximation of $\mathcal{N}(t')$, M is a direct summand of $\mathcal{M}(t')$ by Proposition 6.2, and hence $\mathrm{MCM}\mu(\mathcal{N}(t')) = \mu(M) \leq \mu(\mathcal{M}(t')) \leq n$. \square

Example 6.6. The Betti numbers $\beta_i(M) = \dim \mathrm{Tor}_i^A(M, A/\mathfrak{m}_A)$ are well known upper semi-continuous invariants of finite modules over local rings. By Theorem 6.5 the induced invariants $\mathrm{MCM}\beta_i$, $\mathrm{MCM}\beta'_i$, $\mathrm{FID}\beta_i$ and $\mathrm{FID}\beta'_i$ are upper semi-continuous too.

We now consider some invariants defined in terms of Cohen-Macaulay approximation. If $h : S \rightarrow T$ is in one of the categories of local Cohen-Macaulay maps, a map $\partial : D \rightarrow D'$ of objects in \mathbf{D}_h is said to be *minimal* if $k \otimes_T \partial = 0$. Any module D in \mathbf{D}_h is isomorphic to some $\omega_h^{\oplus n}$ and $\mathrm{End}_T(\omega_h) \cong T$. Hence if ∂ is not minimal then there is a surjection $D' \rightarrow \omega_h$ inducing a surjection $D \rightarrow \omega_h$. By Corollary 5.7 the ω_h splits off from ∂ . Hence any \mathbf{D}_h -complex is homotopy equivalent to one with all differentials being minimal, which is called a minimal \mathbf{D} -complex.

For any module \mathcal{N} in $\mathbf{mod}_h^{\mathfrak{fl}}$ over \mathbf{xCM} we choose a minimal MCM-approximation $\mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N}$ and a minimal $\hat{\mathbf{D}}^{\mathfrak{fl}}$ -hull $\mathcal{N} \rightarrow \mathcal{L}' \rightarrow \mathcal{M}'$ which exist by Corollary 6.3. Spliced with a minimal \mathbf{D} -resolution of \mathcal{L} and a minimal \mathbf{D} -coresolution of \mathcal{M}' we obtain complexes ${}^{-}C^*(\mathcal{N})$, ${}^{+}C^*(\mathcal{N})$ and $D^*(\mathcal{N})$, as defined in Section 2.2, where no differential has any ω_h -summand. We call such choices of these complexes for minimal. They are unique:

Lemma 6.7. *Suppose h is in \mathbf{hCM} , \mathbf{ICM} or \mathbf{cCM} and (h, \mathcal{N}) is in $\mathbf{mod}_h^{\mathfrak{fl}}$. Then minimal choices of ${}^{-}C^*(\mathcal{N})$, ${}^{+}C^*(\mathcal{N})$ and $D^*(\mathcal{N})$ exist and are unique up to non-canonical isomorphisms.*

Proof. Minimal choices ${}^{+}C_1^*$ and ${}^{+}C_2^*$ of coresolutions for \mathcal{N} are by Lemma 2.2 homotopic through chain maps α and β starting with an isomorphism $\mathcal{L}'_1 \cong \mathcal{L}'_2$. If ρ_i are homotopies with $\beta\alpha - \mathrm{id} = \partial\rho_1 + \rho_1\partial$ and $\alpha\beta - \mathrm{id} = \partial\rho_2 + \rho_2\partial$ then tensoring down by $k \otimes_T -$ makes the right hand side of these identities equal to zero by the minimality of the complexes. Hence $\beta\alpha$ and $\alpha\beta$ are surjective endomorphisms, i.e. isomorphisms. The same argument applies to ${}^{-}C^*$ and D^* . \square

For each module \mathcal{N} in $\mathbf{mod}_h^{\mathfrak{fl}}$ we fix a minimal \mathbf{D} -complex $(D^*(\mathcal{N}), \partial^*)$ representing \mathcal{N} . Let $\mathcal{L}' = \mathrm{coker} \partial^{-1}$ and $\mathcal{M} = \ker \partial^0$. Put $\mathrm{Syz}_i^{\omega_h} \mathcal{L}' := \mathrm{coker} \{\partial^{-i-1} : D^{-i-1} \rightarrow D^{-i}\}$ for $i \geq 0$ and $\mathrm{Syz}_i^{\omega_h} \mathcal{M} := \ker \{\partial^i : D^i \rightarrow D^{i+1}\}$ for $i \geq 0$. For any finite T -module \mathcal{N} let the ω_h -rank of \mathcal{N} , denoted $\omega_h\text{-rk}(\mathcal{N})$, be the largest number n with $\omega_h^{\oplus n} \oplus \mathcal{N}' \cong \mathcal{N}$ for some T -module \mathcal{N}' . Since $\mathrm{End}_T(\omega_h) \cong T$ is a local ring, this is a well behaved invariant, cf. [43, Sec. 1.1].

Definition 6.8. Suppose h is in \mathbf{hCM} , \mathbf{ICM} or \mathbf{cCM} and \mathcal{N} is in $\mathbf{mod}_h^{\mathfrak{fl}}$. Define the numbers:

- (i) $d_T^i(\mathcal{N}) := \omega_h\text{-rk}(D^i(\mathcal{N}))$ for all i
- (ii) $\nu_i^T(\mathcal{N}) := \omega_h\text{-rk}(\mathrm{Syz}_i^{\omega_h} \mathcal{L}')$ for $i \geq 0$
- (iii) $\gamma_T(\mathcal{N}) := \omega_h\text{-rk}(\mathcal{M})$

The definition gives well defined invariants of \mathcal{N} by Lemma 6.7. In particular we see that $\nu_0^T(\mathcal{N})$ equals $\omega_h\text{-rk}(\mathcal{L}')$. We also notice that $\omega_h\text{-rk}(\mathrm{Syz}_{-i}^{\omega_h} \mathcal{M}) = 0$ for all $i > 0$ by Proposition 6.2.

The same notation is used for the absolute counterparts of these invariants.

Remark 6.9. In the absolute case with A a Gorenstein local ring, the γ_A -invariant is called Auslander's δ invariant and has been studied in particular by S. Ding. One has that $\gamma_A(A/\mathfrak{m}_A^n) \leq 1$ with equality for $n \gg 0$. The smallest number n with $\gamma_A(A/\mathfrak{m}_A^n) = 1$ is called the *index* of A and Ding has given results and conjectures concerning this invariant, cf. [14]. See also [25].

Let (A, \mathfrak{m}, k) be a noetherian local ring and N a finite A -module. Simon and Strooker introduced the *reduced Bass numbers*

$$(6.9.1) \quad \nu_A^i(N) = \dim_k \operatorname{im}\{\operatorname{Ext}_A^i(k, N) \rightarrow H_{\mathfrak{m}}^i(N)\}$$

and in the case A has a canonical module they showed in [43, 2.6 and 3.10] that

$$(6.9.2) \quad \nu_0^A(N) = \nu_A^{\dim A}(N) \quad \text{and} \quad \nu_i^A(N) = \nu_A^{\dim A-i}(L') \quad \text{for } i \geq 0$$

where L' is the minimal D_A -hull of N . The Canonical Element Conjecture and the Monomial Conjecture are equivalent to $\nu_A^{\dim A}(\mathfrak{b}) \neq 0$ (or equivalently $\gamma_A(A/\mathfrak{b}) = 0$) for certain ideals \mathfrak{b} in any Gorenstein local ring A , see [43, 6.4 and 6.6].

Let \mathbf{P} and $\hat{\mathbf{P}}^{\text{fl}}$ denote the FADs of finite projective modules, respectively modules of finite projective dimension over \mathbf{xCM} defined as for the finite type case.

Lemma 6.10. *With notation as above:*

- (i) *The functor $F = \operatorname{Hom}_-(\omega_-, -)$ gives exact equivalences of categories fibred in additive categories $\mathbf{D} \simeq \mathbf{P}$ and $\hat{\mathbf{D}}^{\text{fl}} \simeq \hat{\mathbf{P}}^{\text{fl}}$ over \mathbf{xCM} .*
- (ii) *In particular $d^{-i}(\mathcal{N}) = \beta_i(\operatorname{Hom}_T(\omega_h, \mathcal{L}'))$ for all $i \geq 0$.*
- (iii) *$\operatorname{Hom}_T(D^{*\geq 0}(\mathcal{N}), \omega_h)$ gives a minimal free resolution of $\operatorname{Hom}_T(\mathcal{M}, \omega_h)$. In particular $d^i(\mathcal{N}) = \beta_i(\operatorname{Hom}_T(\mathcal{M}, \omega_h))$ for all $i \geq 0$.*

Proof. (i) and (ii) are the local variants of Lemma 5.4. Breaking $\mathcal{M} \hookrightarrow D^{*\geq 0}(\mathcal{N})$ into short exact sequences, (iii) follows from Corollary 5.7. \square

Corollary 6.11. *Let $h : S \rightarrow T$ be a finite type Cohen-Macaulay map and suppose \mathcal{N} is a T -module in $\mathbf{mod}_h^{\text{fl}}$. Then $d^i(\mathcal{N}(t))$ are upper semi-continuous functions in $t \in \operatorname{Spec} T$ for all i .*

Proof. This follows from Theorem 6.5 and Lemma 6.10. \square

Remark 6.12. The invariants ν_0 and γ are not semi-continuous either way, see Example 7.4. Moreover, let \mathcal{L} be in $\hat{\mathbf{D}}_h$ with $L = \mathcal{L} \otimes_S k$. If $\rho_0 : \omega_A \rightarrow L$ is a direct summand, then ρ_0 lifts to a map $\rho : \omega_h \rightarrow \mathcal{L}$, but no lifting ρ of ρ_0 has to split, even if A is a regular ring.

Remark 6.13. One can also define functions of the base. E.g. if $\varphi : \operatorname{Spec} T \rightarrow \operatorname{Spec} S$ denotes the map induced from h and $\mu(\mathcal{N}(t))$ is an upper semi-continuous function of $t \in \operatorname{Spec} T$, then $\varphi_*\mu(\mathcal{N})$ defined by

$$(6.13.1) \quad \varphi_*\mu(\mathcal{N})(s) = \sup_{t \in \varphi^{-1}(s)} \mu(\mathcal{N}(t))$$

is an upper semi-continuous function in $s \in \operatorname{Spec} S$ since φ is an open map.

7. THE FUNDAMENTAL MODULE OF A COHEN-MACAULAY MAP

Example 7.1. Let (A, \mathfrak{m}_A, k) be a Cohen-Macaulay local ring with canonical module ω_A . Let $(G_*, d_*) \rightarrow k$ be a minimal A -free resolution of k and put $\operatorname{Syz}_i = \operatorname{Syz}_i^A k = \operatorname{coker} d_{i+1}$. Suppose $\dim A = d \geq 2$. There are connecting isomorphisms $\operatorname{Ext}_A^1(\operatorname{Syz}_{d-1}, \omega_A) \cong \operatorname{Ext}_A^2(\operatorname{Syz}_{d-2}, \omega_A) \cong \dots \cong \operatorname{Ext}_A^d(k, \omega_A)$ which is isomorphic to k by duality theory. To $1 \in k$ there is hence a non-split short exact sequence

$$(7.1.1) \quad \theta : 0 \rightarrow \omega_A \longrightarrow E_A \xrightarrow{\pi} \operatorname{Syz}_{d-1}^A k \rightarrow 0$$

with E_A uniquely defined up to non-canonical isomorphism. We call E_A for the *fundamental module* of A . We claim that E_A is a maximal Cohen-Macaulay module which implies that (7.1.1) is the minimal MCM approximation of $\text{Syz}_{d-1}^A k$. If we apply $\text{Hom}_A(-, \omega_A)$ to (7.1.1) we obtain the exact sequence

$$(7.1.2) \quad 0 \rightarrow \text{Hom}_A(\text{Syz}_{d-1}^A k, \omega_A) \rightarrow \text{Hom}_A(E_A, \omega_A) \rightarrow \text{End}_A(\omega_A) \xrightarrow{\partial} \text{Ext}_A^1(\text{Syz}_{d-1}^A k, \omega_A) \rightarrow \text{Ext}_A^1(E_A, \omega_A) \rightarrow 0$$

We have $\partial(\text{id}) = \theta$ so ∂ is surjective and $\text{Ext}_A^1(E_A, \omega_A) = 0$. By duality theory (e.g. [10, 3.5.11]) this excludes the possibility $\text{depth } E_A = d - 1$ and we conclude that E_A is a maximal Cohen-Macaulay module. If N is a Cohen-Macaulay module of codimension c we denote $\text{Ext}_A^c(N, \omega_A)$ by N^\vee . Since $\text{End}_A(\omega_A) \cong A$ we get from (7.1.2) a short exact sequence:

$$(7.1.3) \quad 0 \rightarrow \text{Hom}_A(\text{Syz}_{d-1}^A k, \omega_A) \rightarrow E_A^\vee \rightarrow \mathfrak{m}_A \rightarrow 0$$

Since $\text{Ext}_A^i(k, \omega_A) = 0$ for $i \neq d$, $0 \rightarrow G_0^\vee \rightarrow \dots \rightarrow G_{d-2}^\vee \rightarrow \text{Hom}_A(\text{Syz}_{d-1}^A k, \omega_A) \rightarrow 0$ is a finite ω_A -resolution and so (7.1.3) gives the minimal MCM approximation of the maximal ideal. Auslander introduced the fundamental module in the case $d = 2$, see [6].

We can make a relative version of the fundamental module in much the same way. Let $(^2)\Delta : \text{CM} \rightarrow \text{CM}$ be the morphism of fibred categories over NR defined by taking the CM map $h : S \rightarrow T$ to the composition $h^{(2)}$ of h with $\iota = 1 \otimes \text{id}_T : T \rightarrow T \otimes_S T$ and taking a morphism $(g, f) : h_1 \rightarrow h_2$ to the composition of two cocartesian squares $(g, f^{\otimes 2})$ as follows:

$$(7.1.4) \quad \begin{array}{ccc} S_1 & \xrightarrow{h_1} & T_1 \\ g \downarrow & & \downarrow f \\ S_2 & \xrightarrow{h_2} & T_2 \end{array} \mapsto \begin{array}{ccccc} S_1 & \xrightarrow{h_1} & T_1 & \xrightarrow{\iota} & T_1 \otimes_{S_1} T_1 \\ g \downarrow & & \downarrow f & & \downarrow f^{\otimes 2} \\ S_2 & \xrightarrow{h_2} & T_2 & \xrightarrow{\iota} & T_2 \otimes_{S_2} T_2 \end{array}$$

There is also a functor $\Delta : \text{CM} \rightarrow \text{CM}$ defined by mapping (g, f) to the rightmost cocartesian square $(f, f^{\otimes 2})$, but it doesn't commute with the forgetful functor $\text{CM} \rightarrow \text{NR}$. Let ${}^d\text{CM}$ denote the full subcategory of CM maps of pure relative dimension d . Then ${}^d\text{CM}$ is a fibred subcategory of CM over NR and $(^2)\Delta$ and Δ restricts to a morphism $(^2)\Delta : {}^d\text{CM} \rightarrow {}^{2d}\text{CM}$ over NR and a functor $\Delta : {}^d\text{CM} \rightarrow {}^d\text{CM}$.

Let $h : S \rightarrow T$ be a finite type CM map of pure relative dimension $d \geq 2$. Consider P in P_h (see Lemma 5.4) as a $T^{\otimes 2}$ -module by pullback along the multiplication map $\mu : T^{\otimes 2} \rightarrow T$. By Corollary 2.9 $E = \text{Ext}_{T^{\otimes 2}}^d(P, \omega_\iota)$ is flat and finite as T -module, i.e. T -projective. Let P^* denote $\text{Hom}_T(P, T)$. By Corollary 2.9

$$(7.1.5) \quad \text{End}_T(E) \cong \text{Ext}_{T^{\otimes 2}}^d(P, \omega_\iota) \otimes E^* \cong \text{Ext}_{T^{\otimes 2}}^d(P, \omega_\iota \otimes E^*).$$

Combined with the connecting isomorphisms the identity in $\text{End}_T(E)$ corresponds to a canonical extension of $T^{\otimes 2}$ -modules:

$$(7.1.6) \quad 0 \rightarrow \omega_\iota \otimes_T \text{Ext}_{T^{\otimes 2}}^d(P, \omega_\iota)^* \rightarrow E_h(P) \rightarrow \text{Syz}_{d-1}^{T^{\otimes 2}} P \rightarrow 0.$$

Let ${}^d\text{P}$ and ${}^d\text{MCM}$ denote the restriction of P and MCM to fibred categories over ${}^d\text{CM}$.

Proposition 7.2. *Let $d \geq 2$. The association $(h, P) \mapsto E_h(P)$ in (7.1.6) induces*

- (i) *a functor $E : {}^d\text{P} \rightarrow {}^d\text{MCM}/{}^d\text{P}$ which preserves cocartesian maps and lifts the functor $\Delta : {}^d\text{CM} \rightarrow {}^d\text{CM}$, and*
- (ii) *a morphism $(^2)E : {}^d\text{P} \rightarrow {}^{2d}\text{MCM}/{}^{2d}\text{P}$ of fibred categories over NR which lifts $(^2)\Delta : {}^d\text{CM} \rightarrow {}^{2d}\text{CM}$.*

Proof. As an extension of T -flat modules, $E_h(P)$ is T -flat. Applying $\text{Hom}_{T^{\otimes 2}}(-, \omega_\iota)$ to (7.1.6) with $E = \text{Ext}_{T^{\otimes 2}}^d(P, \omega_\iota)$ and $\text{Syz}_{d-1} = \text{Syz}_{d-1}^{T^{\otimes 2}}P$ gives an exact sequence

$$(7.2.1) \quad \begin{aligned} 0 \rightarrow \text{Hom}_{T^{\otimes 2}}(\text{Syz}_{d-1}, \omega_\iota) &\rightarrow \text{Hom}_{T^{\otimes 2}}(E_h(P), \omega_\iota) \rightarrow \text{End}_{T^{\otimes 2}}(\omega_\iota) \otimes_T E \xrightarrow{\partial} \\ &\text{Ext}_{T^{\otimes 2}}^1(\text{Syz}_{d-1}, \omega_\iota) \rightarrow \text{Ext}_{T^{\otimes 2}}^1(E_h(P), \omega_\iota) \rightarrow 0 \end{aligned}$$

by Corollary 2.9 and duality theory. In particular there is a canonical isomorphism $\text{Hom}_{T^{\otimes 2}}(\omega_\iota \otimes_T E^*, \omega_\iota) \cong \text{End}_{T^{\otimes 2}}(\omega_\iota) \otimes_T E$. We have that $\text{End}_{T^{\otimes 2}}(\omega_\iota)$ is canonically isomorphic to $T^{\otimes 2}$ and $\partial(t \otimes \xi) = \mu(t) \text{Syz}_{d-1}^{T^{\otimes 2}}(\xi) \in \text{Ext}_{T^{\otimes 2}}^1(\text{Syz}_{d-1}, \omega_\iota)$ where $\text{Syz}_{d-1}^{T^{\otimes 2}}$ is the composition of the connecting isomorphisms. So ∂ is surjective and $\text{Ext}_{T^{\otimes 2}}^1(E_h(P), \omega_\iota) = 0$ by Corollary 2.9. It follows that all fibres of $E_h(P)$ are MCM modules and so $E_h(P)$ is in MCM_ι and (7.1.6) is an MCM_ι -approximation of $\text{Syz}_{d-1}^{T^{\otimes 2}}P$. Let \mathcal{I}_h denote the kernel of $\mu : T^{\otimes 2} \rightarrow T$ and $(-)^{\vee} = \text{Hom}_{T^{\otimes 2}}(-, \omega_\iota)$. From (7.2.1) we get another MCM_ι -approximation

$$(7.2.2) \quad 0 \rightarrow \text{Hom}_{T^{\otimes 2}}(\text{Syz}_{d-1}^{T^{\otimes 2}}P, \omega_\iota) \longrightarrow E_h(P)^{\vee} \longrightarrow \mathcal{I}_h \otimes_T \text{Ext}_{T^{\otimes 2}}^d(P, \omega_\iota) \rightarrow 0.$$

Dualising (7.2.2) induces (7.1.6) since $E_h(P) \cong E_h(P)^{\vee\vee}$ and $\text{Hom}_{T^{\otimes 2}}(\mathcal{I}_h, \omega_\iota) \cong \omega_\iota$. By Theorem 5.1 the image of $E_h(P)^{\vee}$ in $\text{MCM}_\iota/\mathcal{D}_\iota$ is functorial in the $T^{\otimes 2}$ -module $\mathcal{I}_h \otimes E$ which again is contravariantly functorial in P . Since $(-)^{\vee}$ induces an equivalence

$$(7.2.3) \quad \vee : \text{MCM}_\iota/\mathcal{P}_\iota \xleftarrow{\cong} \text{MCM}_\iota^{\text{op}}/\mathcal{D}_\iota^{\text{op}} : \vee$$

we conclude that $E_h(P)$ is functorial in MCM/\mathcal{P} by our functorial choice of extension. This gives (i) and (ii). \square

Corollary 7.3. *For any Cohen-Macaulay map $h : S \rightarrow T$ of pure relative dimension $d \geq 2$ there is a finite $T^{\otimes 2}$ -module $E_h = E_h(T)$ which is faithfully flat along $\iota : T \rightarrow T^{\otimes 2}$ with all fibres being maximal Cohen-Macaulay modules. The association $h \mapsto E_h$ defines a functor ${}^d\text{CM} \rightarrow {}^d\text{MCM}/{}^d\mathcal{P}$ lifting $\Delta : {}^d\text{CM} \rightarrow {}^d\text{CM}$. In particular E_h gives MCM_ι -approximations*

$$0 \rightarrow \omega_\iota \longrightarrow E_h \longrightarrow \text{Syz}_{d-1}^{T^{\otimes 2}}T \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow \text{Hom}_{T^{\otimes 2}}(\text{Syz}_{d-1}^{T^{\otimes 2}}T, \omega_\iota) \longrightarrow E_h^{\vee} \longrightarrow \mathcal{I}_h \rightarrow 0$$

where \mathcal{I}_h is the kernel of the multiplication map $T^{\otimes 2} \rightarrow T$.

Proof. This follows from Proposition 7.2 and (7.2.2) once we have proved the natural isomorphism $\text{Ext}_{T^{\otimes 2}}^d(T, \omega_\iota) \cong T$. Choose a surjection of S -algebras $P \rightarrow T$ with $P = S[t_1, \dots, t_N]$. Recall that ω_ι can be given as $\text{Ext}_{P \otimes T}^{N-d}(T^{\otimes 2}, \omega_{P \otimes T/T})$ where $\omega_{P \otimes T/T} = \wedge^N \Omega_{P \otimes T/T}$. There is a change of rings spectral sequence

$$(7.3.1) \quad E_2^{p,q} = \text{Ext}_{T^{\otimes 2}}^q(T, \text{Ext}_{P \otimes T}^p(T^{\otimes 2}, \omega_{P \otimes T/T})) \Rightarrow \text{Ext}_{P \otimes T}^{p+q}(T, \omega_{P \otimes T/T})$$

which by Corollary 2.9 and duality theory collapses to the canonical isomorphism

$$(7.3.2) \quad \text{Ext}_{T^{\otimes 2}}^d(T, \text{Ext}_{P \otimes T}^{N-d}(T^{\otimes 2}, \omega_{P \otimes T/T})) \cong \text{Ext}_{P \otimes T}^N(T, \omega_{P \otimes T/T}).$$

By [12, 3.5.6] $\text{Ext}_{P \otimes T}^N(T, \omega_{P \otimes T/T})$ is canonically isomorphic to $\omega_{T/T} = T$ as $T^{\otimes 2}$ -module. \square

We call the module E_h given in Corollary 7.3 for the *fundamental module* of the Cohen-Macaulay map h .

Example 7.4. Let k be an algebraically closed field and A a finite type k -algebra which is Cohen-Macaulay of pure dimension 2. Then the fundamental module $E = E_h$ of $h : k \rightarrow A$ is the maximal Cohen-Macaulay approximation of $I = \ker\{A^{\otimes 2} \rightarrow A\}$ in $\mathbf{mod}_\ell^{\text{fl}}$;

$$(7.4.1) \quad 0 \rightarrow \omega_h \otimes A \rightarrow E \rightarrow I \rightarrow 0$$

where $\iota = 1 \otimes \text{id} : A \rightarrow A^{\otimes 2}$ and $\omega_h \cong \omega_A$. Let t in $\text{Spec } A^{\otimes 2}$ be a k -point, and $t_i \in \text{Spec } A$ be the image of t by the i^{th} projection. Let A_i denote A localised at t_i . Let \mathfrak{m}_i be the maximal ideal in A_i . Localising gives a local Cohen-Macaulay map $\iota_{\mathfrak{p}_t} : A_2 \rightarrow (A^{\otimes 2})_{\mathfrak{p}_t}$ and a module $E_{\mathfrak{p}_t}$ in $\mathbf{MCM}_{\iota_{\mathfrak{p}_t}}$. Let $E(t)$ denote base change of $E_{\mathfrak{p}_t}$ to $k(t_2)$. If $t_1 = t_2$ then $I(t) \cong \mathfrak{p}_1$ and $E(t)$ equals the fundamental module E_{A_1} of (7.1.1). If $t_1 = t_2$ is singular, then $\omega\text{-rk}(E(t)) = 0$ while if $t_1 = t_2$ is regular then $E(t) \cong A_1^{\oplus 2}$. If $t_1 \neq t_2$ then $I(t) \cong A_1 \cong E_{A_1}$ and $E(t) \cong A_1^{\oplus 2}$. This shows that $\gamma(I)(t)$ is *not* upper semi-continuous as the d^i -invariants are.

In particular, if A equals $k[x, y, z]/(x^{n+1} - yz)$ with a 2-dimensional rational double point at $\mathfrak{m}_0 = (x, y, z)$, similar considerations give the following table of invariants (note that $\nu_1 = d^{-1}$):

$\iota : A \rightarrow A^{\otimes 2}$	γ	ν_1	d^0	ν_0	$I(t)$
$t_1 = t_2 = 0$ singular point	0	1	4	1	$\mathfrak{m}_0 A_{\mathfrak{m}_0}$
$t_1 = t_2$ non-singular point	2	1	2	0	$\mathfrak{m}_1 A_1$
$t_1 \neq t_2$	1	0	1	1	A_1

8. DEFORMATION FUNCTORS AND COHOMOLOGY

We extend the Cohen-Macaulay approximation over henselian local rings to deformations and obtain maps between the associated deformation functors. We also introduce the appropriate André-Quillen cohomology and links the various cohomologies in a fundamental long-exact sequence.

Fix an object $\xi = (h : S \rightarrow T, \mathcal{N})$ in \mathbf{mod}^{fl} over \mathbf{H} . A *deformation* of ξ is a cocartesian morphism $\alpha_1 : \xi_1 \rightarrow \xi$ in \mathbf{mod}^{fl} . A *map of deformations* $\alpha_1 \rightarrow \alpha_2$ is a cocartesian morphism $\varphi : \xi_1 \rightarrow \xi_2$ in \mathbf{mod}^{fl} such that $\alpha_2 \varphi = \alpha_1$. Deformations and maps of deformations are objects and arrows in the comma category $\mathbf{Def}_\xi := \mathbf{mod}_{\text{coca}}^{\text{fl}}/\xi$ which is fibred in groupoids over the comma category \mathbf{H}/S , see Lemma 3.1 and the proceeding comments. Let the *deformation functor* $\mathbf{Def}_\xi : \mathbf{H}/S \rightarrow \mathbf{Sets}$ be the functor corresponding to the associated groupoid of sets $\overline{\mathbf{Def}}_\xi$. The comma category $\mathbf{Def}_h := \mathbf{hCM}/h$ of deformations of $h : S \rightarrow T$ is also fibred in groupoids over \mathbf{H}/S and we have an obvious factorisation $\mathbf{Def}_\xi \rightarrow \mathbf{hCM}/h \rightarrow \mathbf{H}/S$ which makes \mathbf{Def}_ξ fibred in groupoids over \mathbf{hCM}/h . To ease readability (and by abuse of notation) we put $\mathbf{Def}_{(T, \mathcal{N})}(S_1) = \mathbf{Def}_\xi(S_1 \rightarrow S)$ and $\mathbf{Def}_T(S_1) = \mathbf{Def}_h(S_1 \rightarrow S)$. We also write a *deformation of* (T, \mathcal{N}) meaning a deformation of ξ and likewise in similar situations.

For each object $\xi_i = (h_i, \mathcal{N}_i)$ in \mathbf{mod}^{fl} over \mathbf{H} we choose a minimal MCM-approximation $\pi_i : \mathcal{L}_i \rightarrow \mathcal{M}_i \xrightarrow{\pi_i} \mathcal{N}_i$ and a minimal $\hat{\mathbf{D}}^{\text{fl}}$ -hull $\iota_i : \mathcal{N}_i \xrightarrow{\iota_i} \mathcal{L}'_i \rightarrow \mathcal{M}'_i$ which exist by Corollary 5.7 and Corollary 6.3. For each deformation $\alpha_i : \xi_i \rightarrow \xi_0$ we choose extensions to commutative diagrams of deformations

$$(8.0.2) \quad \begin{array}{ccccc} \mathcal{L}_i & \longrightarrow & \mathcal{M}_i & \xrightarrow{\pi_i} & \mathcal{N}_i \\ \lambda_i \downarrow & & \mu_i \downarrow & & \nu_i \downarrow \\ \mathcal{L}_0 & \longrightarrow & \mathcal{M}_0 & \xrightarrow{\pi_0} & \mathcal{N}_0 \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathcal{N}_i & \xrightarrow{\iota_i} & \mathcal{L}'_i & \longrightarrow & \mathcal{M}'_i \\ \nu_i \downarrow & & \lambda'_i \downarrow & & \mu'_i \downarrow \\ \mathcal{N}_0 & \xrightarrow{\iota_0} & \mathcal{L}'_0 & \longrightarrow & \mathcal{M}'_0 \end{array}$$

as follows: By Corollary 6.3 a base change of π_i by $h_i \rightarrow h_0$ gives a minimal MCM_{h_0} -approximation $\mathcal{M}_i^\# \rightarrow \mathcal{N}_i^\# \xrightarrow{\simeq} \mathcal{N}_0$. By minimality it is isomorphic to π_0 . Choose an isomorphism. Let μ_i be the composition $\mathcal{M}_i \rightarrow \mathcal{M}_i^\# \xrightarrow{\simeq} \mathcal{M}_0$. It is cocartesian. Do similarly for the $\hat{\mathbf{D}}^{\text{fl}}$ -hull. Let these choices be fixed.

The diagrams in (8.0.2) will be called an **MCM-approximation** (denoted π_*) respectively a $\hat{\mathbf{D}}^{\text{fl}}$ -hull (denoted ι_*) of ν_i (this terminology can be justified).

Definition 8.1. There are four maps

$$\sigma_X : \text{Def}_{(h_0, \mathcal{N}_0)} \longrightarrow \text{Def}_{(h_0, X)} \text{ of functors } \mathbf{H}/S_0 \longrightarrow \mathbf{Sets}$$

where X can be $\mathcal{M}_0, \mathcal{L}_0, \mathcal{L}'_0$ and \mathcal{M}'_0 given by $[(h_i \rightarrow h_0, \nu_i)] \mapsto [(h_i \rightarrow h_0, x)]$ for x equal to $\mu_i, \lambda_i, \lambda'_i$ and μ'_i in (8.0.2) respectively.

The following lemma implies that these maps are well defined and independent of choices.

Lemma 8.2. *Given two deformations $\nu_{ij} : \mathcal{N}_{ij} \rightarrow \mathcal{N}_{0j}$ ($j = 1, 2$) in mod^{fl} over $h_{ij} \rightarrow h_{0j}$ in \mathbf{hCM} and MCM-approximations π_{*j} (respectively $\hat{\mathbf{D}}^{\text{fl}}$ -hulls ι_{*j}) of ν_{ij} . Suppose we have a map of short exact sequences $\pi_{01} \rightarrow \pi_{02}$ (respectively $\iota_{01} \rightarrow \iota_{02}$) and a map $\alpha : \mathcal{N}_{i1} \rightarrow \mathcal{N}_{i2}$ lifting $\mathcal{N}_{01} \rightarrow \mathcal{N}_{02}$, i.e. such that the following two diagrams of solid arrows are commutative:*

$$\begin{array}{ccccc} & \mathcal{L}_{i2} & \longrightarrow & \mathcal{M}_{i2} & \xrightarrow{\pi_{i2}} & \mathcal{N}_{i2} \\ & \nearrow \lambda_{i2} & & \nearrow \mu_{i2} & & \nearrow \nu_{i2} \\ \mathcal{L}_{i1} & \longrightarrow & \mathcal{M}_{i1} & \xrightarrow{\pi_{i1}} & \mathcal{N}_{i1} \\ \downarrow \lambda_{i1} & & \downarrow \mu_{i1} & & \downarrow \nu_{i1} \\ \mathcal{L}_{02} & \longrightarrow & \mathcal{M}_{02} & \xrightarrow{\pi_{02}} & \mathcal{N}_{02} \\ \uparrow \beta & & \uparrow \alpha & & \uparrow \nu_{01} \\ \mathcal{L}_{01} & \longrightarrow & \mathcal{M}_{01} & \xrightarrow{\pi_{01}} & \mathcal{N}_{01} \end{array} \quad \begin{array}{ccccc} & \mathcal{N}_{i2} & \xrightarrow{\iota_{i2}} & \mathcal{L}'_{i2} & \longrightarrow & \mathcal{M}'_{i2} \\ & \nearrow \nu_{i2} & & \nearrow \lambda'_{i2} & & \nearrow \mu'_{i2} \\ \mathcal{N}_{i1} & \xrightarrow{\iota_{i1}} & \mathcal{L}'_{i1} & \longrightarrow & \mathcal{M}'_{i1} \\ \downarrow \nu_{i1} & & \downarrow \lambda'_{i1} & & \downarrow \mu'_{i1} \\ \mathcal{N}_{02} & \xrightarrow{\iota_{02}} & \mathcal{L}'_{02} & \longrightarrow & \mathcal{M}'_{02} \\ \uparrow \beta' & & \uparrow \alpha' & & \uparrow \nu_{01} \\ \mathcal{N}_{01} & \xrightarrow{\iota_{01}} & \mathcal{L}'_{01} & \longrightarrow & \mathcal{M}'_{01} \end{array}$$

Then there exist maps $\gamma : \mathcal{M}_{i1} \rightarrow \mathcal{M}_{i2}$ and $\gamma' : \mathcal{L}'_{i1} \rightarrow \mathcal{L}'_{i2}$ such that the induced left (respectively right) diagram commutes. If α is cocartesian, so are γ and γ' .

Proof. Consider the MCM-approximation case. By applying base changes to the front diagram, we can reduce the problem to the case $h_{i1} \rightarrow h_{01}$ equals $h_{i2} \rightarrow h_{02}$. Then, by Corollary 5.7, there is a lifting $\gamma_1 : \mathcal{M}_{i1} \rightarrow \mathcal{M}_{i2}$ of α . We would like to adjust γ_1 so that it lifts β too. We have that $\mu_{i2}\gamma_1 - \beta\mu_{i1}$ factors through \mathcal{L}_{02} by a map $\tau_i : \mathcal{M}_{i1} \rightarrow \mathcal{L}_{02}$ which induces a unique map $\tau_0 : \mathcal{M}_{01} \rightarrow \mathcal{L}_{02}$ since μ_{i1} is cocartesian. If $\mathcal{D}_* \rightarrow \mathcal{L}_{i2}$ is a finite D-resolution, then base change gives a finite D-resolution $\mathcal{D}_*^\# \rightarrow \mathcal{L}_{02}$ and τ_0 lifts to a $\sigma_0 : \mathcal{M}_{01} \rightarrow \mathcal{D}_0^\#$ by Corollary 5.7. By Corollary 2.9 there is a $\sigma : \mathcal{M}_{i1} \rightarrow \mathcal{D}_0$ lifting σ_0 and subtracting the induced map $\mathcal{M}_{i1} \rightarrow \mathcal{M}_{i2}$ from γ_1 gives our desired γ . If α is an isomorphism so is γ by minimality of the approximations π_{ij} . The argument for the $\hat{\mathbf{D}}^{\text{fl}}$ -case is similar. \square

Remark 8.3. There are maps of fibred categories inducing the maps σ_X in Definition 8.1. Two maps $\alpha, \beta : (h_1, \mathcal{N}_1) \rightarrow (h_2, \mathcal{N}_2)$ in $\text{Def}_{(h_0, \mathcal{N}_0)}$ are stably equivalent if $h_1 = h_2$ and $\alpha - \beta$ factors through an object in \mathbf{D} . Let $\underline{\text{Def}}_{(h_0, \mathcal{N}_0)}$ denote the resulting quotient category which is fibred over \mathbf{hCM}/h_0 and over \mathbf{H}/S_0 . Then Lemma 8.2 implies that there are well defined maps of categories fibred in groupoids $\sigma_X : \underline{\text{Def}}_{(h_0, \mathcal{N}_0)} \rightarrow \underline{\text{Def}}_{(h_0, X)}$ for X equal to $\mathcal{M}_0, \mathcal{L}_0, \mathcal{L}'_0$ and \mathcal{M}'_0 . The associated map of functors is σ_X . Note that $\underline{\text{Def}}_{(h_0, \mathcal{N}_0)}$ is different from $\underline{\text{mod}}_{\text{coCa}}^{\text{fl}}/(h_0, \mathcal{N}_0)$. Stably isomorphic modules will in general have different deformation functors. E.g. let $N = A \oplus \omega_A$. If A is not Gorenstein, then one can have $\text{Ext}_A^1(N, N) \neq 0$. But in the stable category N is isomorphic to A which is infinitesimally rigid.

We have the following reformulation. To $(h : S \rightarrow T, \mathcal{N})$ in mod^{fl} consider $\Gamma = T \oplus \mathcal{N}$ as a graded S -algebra with T in degree 0 and \mathcal{N} in degree 1. A deformation of graded algebras $\Gamma_1 \rightarrow \Gamma$ over $S_1 \rightarrow S$ in H/S is equivalent to a deformation (T_1, \mathcal{N}_1) of (T, \mathcal{N}) . More generally, given a homogeneous morphism of \mathbb{Z} -graded rings $S \rightarrow T$ and a graded T -module M , there are *André-Quillen cohomology groups of graded algebras* ${}_0\text{H}^i(S, T, M) = \text{H}^i \text{Hom}_T^{\text{gr}}(L_{T/S}^{\text{gr}}, M)$. Here $L_{T/S}^{\text{gr}}$ is the graded cotangent complex defined as $\Omega_{P/S} \otimes_P T$ where $P = P_S(T)$ is a graded simplicial degree-wise free S -algebra resolution of T and $\Omega_{P/S}$ denotes the Kähler differentials, see [30, IV] for more details (in a more general situation). See also [34].

Definition 8.4. Given graded ring homomorphisms $h : S \rightarrow T$ and $p : S_1 \rightarrow S$. Assume p is surjective. A *lifting* of h ('of T ') along p ('to S_1 ') is a commutative diagram of graded ring homomorphisms

$$\begin{array}{ccc} T & \xleftarrow{q} & T_1 \\ h \uparrow & & \uparrow h_1 \\ S & \xleftarrow{p} & S_1 \end{array} \quad \text{with } q \otimes S : T_1 \otimes_{S_1} S \cong T \text{ and } \text{Tor}_1^{S_1}(T_1, S) = 0.$$

Two liftings T_1 and T'_1 of T to S_1 are equivalent if there is a graded S_1 -algebra isomorphism $T_1 \cong T'_1$ commuting with q and q' .

There is an obstruction theory for liftings of graded algebras in terms of graded André-Quillen cohomology groups.

Proposition 8.5 ([30, 34]). *Given graded ring homomorphisms $S \rightarrow T$ and $p : S_1 \rightarrow S$ with p surjective and $I^2 = 0$ for $I = \ker p$.*

- (i) *There exists an element $\text{ob}(p, T) \in {}_0\text{H}^2(S, T, T \otimes I)$ which is natural in p such that $\text{ob}(p, T) = 0$ if and only if there exists a lifting of T to S_1 .*
- (ii) *If $\text{ob}(p, T) = 0$ then the set of equivalence classes of liftings of T to S_1 is a torsor for ${}_0\text{H}^1(S, T, T \otimes I)$ which is natural in p .*

The element $\text{ob}(p, T)$ is called the *obstruction class* of (p, T) . If the rings and modules are concentrated in degree 0 this equals the ungraded case and the cohomology groups equals the ungraded André-Quillen cohomology $\text{H}^*(S, T, T \otimes I)$.

Definition 8.6. Given a lifting diagram of ungraded ring homomorphisms as in Definition 8.4 and a T -module N . Then a *lifting* of N to T_1 is a T_1 -module N_1 with $\text{Tor}_1^{S_1}(N_1, S) = 0$ and a map $N_1 \rightarrow N$ inducing an isomorphism $N_1 \otimes S \cong N$. Two liftings N_1 and N'_1 of N to T_1 are *equivalent* if there is an isomorphism of T_1 -modules $N_1 \cong N'_1$ commuting with the maps to N .

There is also an obstruction theory for liftings of modules in terms of Ext groups.

Proposition 8.7 ([30, IV 3.1.5]). *Given (ungraded) ring homomorphisms as in Definition 8.4 with $I^2 = 0$ and a T -module N .*

- (i) *There exists an element $\text{ob}(q, N) \in \text{Ext}_T^2(N, N \otimes I)$ which is natural in q such that $\text{ob}(q, N) = 0$ if and only if there exists a lifting of N to T_1 .*
- (ii) *If $\text{ob}(q, N) = 0$ then the set of equivalence classes of liftings of N to T_1 is a torsor for $\text{Ext}_T^1(N, N \otimes I)$ which is natural in q .*

The element $\text{ob}(q, N)$ is called the *obstruction class* of (q, N) . The following long-exact sequence connects these three cohomologies.

Proposition 8.8. *Suppose T is an (ungraded) S -algebra and N is a T -module. Let $\Gamma = T \oplus N$ be the graded S -algebra with T in degree 0 and N in degree 1 and*

let J be a graded Γ -module with graded components $J = J_0 \oplus J_1$ of degree 0 and 1. Then there is a natural long-exact sequence:

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}_T(N, J_1) \rightarrow {}_0\mathrm{Der}_S(\Gamma, J) \rightarrow \mathrm{Der}_S(T, J_0) \xrightarrow{\partial} \\ \mathrm{Ext}_T^1(N, J_1) \rightarrow {}_0\mathrm{H}^1(S, \Gamma, J) \rightarrow \mathrm{H}^1(S, T, J_0) \xrightarrow{\partial} \mathrm{Ext}_T^2(N, J_1) \rightarrow \dots \end{aligned}$$

Proof. To the graded ring homomorphisms $S \rightarrow T \rightarrow \Gamma$ there is a distinguished triangle of transitivity

$$(8.8.1) \quad L_{\Gamma/T/S}^{\mathrm{gr}} : L_{T/S}^{\mathrm{gr}} \otimes_T \Gamma \rightarrow L_{\Gamma/S}^{\mathrm{gr}} \rightarrow L_{\Gamma/T}^{\mathrm{gr}} \rightarrow L_{T/S}^{\mathrm{gr}} \otimes_T \Gamma[1]$$

in the graded derived category of Γ , see [30, IV 2.3.4]. The (standard) simplicial resolution $P_T(\Gamma)$ equals T in degree 0, the (standard) T -free resolution $F_T(N)$ of the T -module N in degree 1, and higher degree terms, see [30, IV 1.3.2.1]. It follows that $\mathrm{Hom}_{\Gamma}^{\mathrm{gr}}(L_{\Gamma/T}^{\mathrm{gr}}, J) = \mathrm{Hom}_T(F_T(N), J_1)$. Since $L_{T/S}^{\mathrm{gr}} = L_{T/S}$ is concentrated in degree 0, $\mathrm{Hom}_{\Gamma}^{\mathrm{gr}}(L_{T/S}^{\mathrm{gr}} \otimes_T \Gamma, J) = \mathrm{Hom}_T(L_{T/S}, J_0)$. \square

Lemma 8.9. *Let $h : S \rightarrow T$ be a finite type Cohen-Macaulay map and let \mathcal{N} be a T -module in $\mathrm{mod}_h^{\mathrm{fl}}$. Let $\mathcal{L} \rightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{N}$ and $\mathcal{N} \xrightarrow{\iota} \mathcal{L}' \rightarrow \mathcal{M}'$ be an MCM_h -approximation and a $\hat{\mathrm{D}}_h^{\mathrm{fl}}$ -hull of \mathcal{N} . Let X_i denote \mathcal{N} , \mathcal{M} and \mathcal{L}' for $i = 0, 1, 2$ respectively, and put $\Gamma_i = T \oplus X_i$. Let I be any S -module. Then there are natural maps of short exact sequences of complexes (see (8.8.1))*

$$\mathrm{Hom}_{\Gamma_0}^{\mathrm{gr}}(L_{\Gamma_0/T/S}^{\mathrm{gr}}, \Gamma_0 \otimes I) \xrightarrow{\pi^*} \mathrm{Hom}_{\Gamma_1}^{\mathrm{gr}}(L_{\Gamma_1/T/S}^{\mathrm{gr}}, \Gamma_0 \otimes I) \xleftarrow{\pi_*} \mathrm{Hom}_{\Gamma_1}^{\mathrm{gr}}(L_{\Gamma_1/T/S}^{\mathrm{gr}}, \Gamma_1 \otimes I)$$

and

$$\mathrm{Hom}_{\Gamma_0}^{\mathrm{gr}}(L_{\Gamma_0/T/S}^{\mathrm{gr}}, \Gamma_0 \otimes I) \xrightarrow{\iota_*} \mathrm{Hom}_{\Gamma_0}^{\mathrm{gr}}(L_{\Gamma_0/T/S}^{\mathrm{gr}}, \Gamma_2 \otimes I) \xleftarrow{\iota^*} \mathrm{Hom}_{\Gamma_2}^{\mathrm{gr}}(L_{\Gamma_2/T/S}^{\mathrm{gr}}, \Gamma_2 \otimes I).$$

The induced maps of graded André-Quillen cohomology

$$\begin{aligned} {}_0\mathrm{H}^n(\pi_*) : {}_0\mathrm{H}^n(S, \Gamma_1, \Gamma_1 \otimes I) &\rightarrow {}_0\mathrm{H}^n(S, \Gamma_1, \Gamma_0 \otimes I) \quad \text{and} \\ {}_0\mathrm{H}^n(\iota^*) : {}_0\mathrm{H}^n(S, \Gamma_2, \Gamma_2 \otimes I) &\rightarrow {}_0\mathrm{H}^n(S, \Gamma_0, \Gamma_2 \otimes I) \end{aligned}$$

are isomorphisms for $n > 0$ and surjections for $n = 0$.

Proof. There is a natural map $L_{\Gamma_1/T/S}^{\mathrm{gr}} \otimes_{\Gamma_1} \Gamma_0 \rightarrow L_{\Gamma_0/T/S}^{\mathrm{gr}}$ of short exact sequences of complexes (cf. [30, II 2.1.1.6]) induced by the graded T -algebra map $\Gamma_1 \rightarrow \Gamma_0$. This gives π^* . Covariance along $\Gamma_1 \otimes I \rightarrow \Gamma_0 \otimes I$ gives π_* . In each (cohomological) degree the rightmost terms are naturally identified with $\mathrm{Hom}_T(L_{T/S}, T \otimes I)$ as in the proof of Proposition 8.8. By Theorem 5.1 and Corollary 2.9 one has $\mathrm{Ext}_T^n(\mathcal{M}, \mathcal{L} \otimes I) = 0$ for $n > 0$ and the ${}_0\mathrm{H}^n(\pi_*)$ -statement follows. The other case is similar. \square

By Lemma 8.9 (and Theorem 5.1) we get induced natural maps for $n > 0$

$$(8.9.1) \quad \sigma_j^n(I) : {}_0\mathrm{H}^n(S, \Gamma_0, \Gamma_0 \otimes_S I) \rightarrow {}_0\mathrm{H}^n(S, \Gamma_j, \Gamma_j \otimes_S I) \text{ for } j = 1, 2 \quad \text{and}$$

$$(8.9.2) \quad \tau_j^n(I) : \mathrm{Ext}_T^n(X_0, X_0 \otimes_S I) \rightarrow \mathrm{Ext}_T^n(X_j, X_j \otimes_S I) \text{ for } j = 1, 2.$$

Example 8.10. By elementary diagram chase Lemma 8.9 gives the following:

- (i) If $\pi^* : \mathrm{Ext}_T^n(\mathcal{N}, \mathcal{N} \otimes I) \rightarrow \mathrm{Ext}_T^n(\mathcal{M}, \mathcal{N} \otimes I)$ is an isomorphism for $n = 1$ and injective for $n = 2$ then $\sigma_1^1(I)$ is an isomorphism and $\sigma_1^2(I)$ is injective.
- (ii) If $\iota_* : \mathrm{Ext}_T^n(\mathcal{N}, \mathcal{N} \otimes I) \rightarrow \mathrm{Ext}_T^n(\mathcal{N}, \mathcal{L}' \otimes I)$ is an isomorphism for $n = 1$ and injective for $n = 2$ then $\sigma_2^1(I)$ is an isomorphism and $\sigma_2^2(I)$ is injective.

9. MAPS OF DEFORMATION FUNCTORS INDUCED BY COHEN-MACAULAY APPROXIMATION

In order to use Artin's Approximation Theorem [3] as extended by D. Popescu [40, 41] we fix an excellent ring Λ (see [19, 7.8.2]). We consider the category of henselian local Λ -algebras in \mathbf{H} , denoted ${}_{\Lambda}\mathbf{H}$. Fibred categories \mathbf{hCM} and $\mathbf{mod}^{\mathbf{fl}}$ over ${}_{\Lambda}\mathbf{H}$ and \mathbf{Def}_h and \mathbf{Def}_{ξ} over ${}_{\Lambda}\mathbf{H}/S$ are defined essentially as in Section 8. Our previous constructions and results are valid in this context as well. In particular deformation functors $\mathbf{Def}_{(T, \mathcal{N})} : {}_{\Lambda}\mathbf{H}/S \rightarrow \mathbf{Sets}$ are defined and the MCM-approximation and $\hat{\mathbf{D}}^{\mathbf{fl}}$ -hull induce maps of deformation functors as in Definition 8.1

Definition 9.1. Let ${}_{\Lambda}\mathbf{A}/k$ denote the subcategory of artin rings in ${}_{\Lambda}\mathbf{H}/k$. Let F and G be set-valued functors on ${}_{\Lambda}\mathbf{H}/k$ (or ${}_{\Lambda}\mathbf{A}/k$) with $\#F(k) = 1 = \#G(k)$. A map $\varphi : F \rightarrow G$ is *smooth* (formally smooth) if the natural map of sets $f_{\varphi} : F(S) \rightarrow F(S_0) \times_{G(S_0)} G(S)$ is surjective for all surjections $\pi : S \rightarrow S_0$ in ${}_{\Lambda}\mathbf{H}/k$ (in ${}_{\Lambda}\mathbf{A}/k$). An element $\nu \in F(R)$ is *versal* if the induced map $\mathrm{Hom}_{{}_{\Lambda}\mathbf{H}/k}(R, -) \rightarrow F$ is smooth and R is algebraic as Λ -algebra. If the map is bijective then ν is universal. An element $\nu \in F(R)$ (or a formal element, i.e. a tower $\{\nu_n\} \in \varprojlim F(R/\mathfrak{m}_R^{n+1})$) is *formally versal* if the induced map $\mathrm{Hom}_{{}_{\Lambda}\mathbf{H}/k}(R, -) \rightarrow F$ of functors restricted to ${}_{\Lambda}\mathbf{A}/k$ is formally smooth. See [4].

Theorem 9.2. Let k be a field, A a Cohen-Macaulay local algebraic k -algebra and N a finite A -module. Let $N \rightarrow L' \rightarrow M'$ be the minimal $\hat{\mathbf{D}}_A$ -hull and $L \rightarrow M \rightarrow N$ the minimal \mathbf{MCM}_A -approximation of N . Consider the map

$$\sigma_{L'} : \mathbf{Def}_{(A, N)} \longrightarrow \mathbf{Def}_{(A, L')} \text{ of functors } {}_{\Lambda}\mathbf{H}/k \longrightarrow \mathbf{Sets}$$

as in Definition 8.1.

- (i) If $\mathrm{Hom}_A(N, M') = 0$ then $\sigma_{L'}$ is injective.
- (ii) If $\mathrm{Ext}_A^1(N, M') = 0$ then $\sigma_{L'}$ is formally smooth.
- (iii) If $\mathrm{Ext}_A^1(N, M') = 0$ and $\mathbf{Def}_{(A, N)}$ has a versal element then $\sigma_{L'}$ is smooth.

The analogous statements hold for $\sigma_L : \mathbf{Def}_{(A, N)} \rightarrow \mathbf{Def}_{(A, L)}$ with $\mathrm{Ext}_A^1(N, M) = 0$ in (i) and $\mathrm{Ext}_A^2(N, M) = 0$ in (ii) and (iii).

Example 9.3. Note that $\mathrm{grade} N \geq 1$ implies condition (i) and $\mathrm{grade} N \geq 2$ implies both condition (i) and (ii).

Proof. (i): Given S in ${}_{\Lambda}\mathbf{H}/k$ and deformations $({}^i h : S \rightarrow {}^i T, {}^i \mathcal{N})$ of (A, N) to S for $i = 1, 2$. Assume that the images $({}^i h, {}^i \mathcal{L}')$ under $\sigma_{L'}$ are isomorphic to $(h : S \rightarrow T, \mathcal{L}')$. Pullback of all these modules along the isomorphisms of h with ${}^i h$ induce deformations over h . We show that the ${}^i \mathcal{N}$ are isomorphic as deformations over h which implies that $\sigma_{L'}$ is injective. Let $S_n = S/\mathfrak{m}_S^{n+1}$, $T_n = T \otimes S_n$ etc.. We construct a tower of isomorphisms $\{\varphi_n : {}^1 \mathcal{N}_n \cong {}^2 \mathcal{N}_n\}$ and conclude by Lemma 6.1 that the deformations are isomorphic. The case $n = 0$ is trivial. Given φ_n and use it to identify the ${}^i \mathcal{N}_n$ and denote them by \mathcal{N}_n . Let $I = \ker\{S_{n+1} \rightarrow S_n\}$. By Proposition 8.7 there exists an element θ in $\mathrm{Ext}_{T_n}^1(\mathcal{N}_n, \mathcal{N}_n \otimes I)$ giving the “difference” of the two deformations of \mathcal{N}_n . But $\mathcal{N}_n \otimes I \cong N \otimes I$ and by the edge map isomorphism of (2.7.1) we get $\mathrm{Ext}_{T_n}^i(\mathcal{N}_n, \mathcal{N}_n \otimes I) \cong \mathrm{Ext}_A^i(N, N) \otimes I$ for all i . If $i > 0$ then $\mathrm{Ext}_A^i(L', L') \cong \mathrm{Ext}_A^i(N, L')$ and $\mathrm{Ext}_A^1(N, N) \rightarrow \mathrm{Ext}_A^1(N, L')$ is injective by assumption. The obtained injective map $p : \mathrm{Ext}_A^1(N, N) \otimes I \rightarrow \mathrm{Ext}_A^1(L', L') \otimes I$ induces a map of the torsor actions in Proposition 8.7 on the liftings of \mathcal{N}_n and of \mathcal{L}'_n to T_{n+1} . Since the $\hat{\mathbf{D}}_{h_{n+1}}^{\mathbf{fl}}$ -hulls of the ${}^i \mathcal{N}_{n+1}$ are isomorphic as deformations, p maps θ to 0 and so $\theta = 0$ and by Proposition 8.7 the ${}^i \mathcal{N}_{n+1}$ are isomorphic by an isomorphism φ_{n+1} compatible with φ_n .

(ii): Let $S \rightarrow \bar{S}$ in ${}_A\mathbf{H}/k$ be surjective with kernel I , $\xi = (h : S \rightarrow T, \mathcal{L}')$ a deformation of (A, L') to S and let $\bar{\xi} = (\bar{h} : \bar{S} \rightarrow \bar{T}, \bar{\mathcal{L}}')$ denote the base change of ξ to \bar{S} . Suppose there is a deformation $(h^{(1)} : \bar{S} \rightarrow T^{(1)}, \bar{\mathcal{N}})$ of (A, N) which $\sigma_{L'}$ maps to $\bar{\xi}$. As above we can assume that $h^{(1)} = \bar{h}$. By induction on the length of S we can assume that $I \cdot \mathfrak{m}_S = 0$. We find that $\text{ob}(T \rightarrow \bar{T}, \bar{\mathcal{N}})$ in Proposition 8.7 maps to $\text{ob}(T \rightarrow \bar{T}, \bar{\mathcal{L}}')$ under $\text{Ext}_A^2(N, N) \otimes I \rightarrow \text{Ext}_A^2(L', L') \otimes I$ which by the assumption is injective. Since \mathcal{L}' lifts $\bar{\mathcal{L}}'$ to T we have $\text{ob}(T \rightarrow \bar{T}, \bar{\mathcal{L}}') = 0$. By Proposition 8.7 there exists a lifting ${}^1\mathcal{N}$ of $\bar{\mathcal{N}}$ to T . If $\sigma_{L'}({}^1\mathcal{N}) = {}^1\mathcal{L}'$ the difference of ${}^1\mathcal{L}'$ and \mathcal{L}' gives by Proposition 8.7 a $\theta \in \text{Ext}_A^1(L', L') \otimes I$. By assumption $\text{Ext}_A^1(N, N) \otimes I$ maps surjectively to $\text{Ext}_A^1(L', L') \otimes I$ and a lifting of θ perturbs ${}^1\mathcal{N}$ to a lifting \mathcal{N} of $\bar{\mathcal{N}}$ with $\sigma_{L'}(\mathcal{N}) = \mathcal{L}'$.

(iii): Let $S \rightarrow \bar{S}$ in ${}_A\mathbf{H}/k$ be surjective with kernel J , $\xi = (h : S \rightarrow T, \mathcal{L}')$ a deformation of (A, L') to S and let $\bar{\xi} = (\bar{h} : \bar{S} \rightarrow \bar{T}, \bar{\mathcal{L}}')$ denote the base change of ξ to \bar{S} . Suppose there is a deformation $(h^{(1)} : \bar{S} \rightarrow T^{(1)}, \bar{\mathcal{N}})$ of (A, N) which $\sigma_{L'}$ maps to $\bar{\xi}$. Again we can assume that $h^{(1)} = \bar{h}$. We will find a deformation \mathcal{N} lifting $\bar{\mathcal{N}}$ such that $\sigma_{L'}(h, \mathcal{N}) = (h, \mathcal{L}')$.

Any S in ${}_A\mathbf{H}/k$ is a direct limit of a filtering system of algebraic A -algebras in ${}_A\mathbf{H}/k$. Since $\text{Def}_{(A, L')}$ is locally of finite presentation (A is algebraic and L' has finite presentation) it is sufficient to prove the lifting property for S algebraic. Since A is excellent, so is S by [19, 7.8.3] and [21, 18.7.6].

Put $S_n = S/\mathfrak{m}_S^n J$, $\mathcal{L}'_n = \mathcal{L}' \otimes_S S_n$, $T_n = T \otimes_S S_n$ and so on. We proceed by induction on n to construct a tower $\{\mathcal{N}_n\}$ of deformations of $\bar{\mathcal{N}}$ inducing $\{\mathcal{L}'_n\}$. Each step is done as in (ii). If $({}^vT, {}^v\mathcal{N}) \in \text{Def}_{(A, N)}(R)$ is a versal element, there is a corresponding tower of maps $\{f_n : R \rightarrow S_n\}$ such that $({}^vT, {}^v\mathcal{N})$ induces $\{(T_n, \mathcal{N}_n)\}$. We obtain the algebra map $f : R \rightarrow {}^*S := \varprojlim S_n$ which induces a deformation $({}^*T, {}^*\mathcal{N})$ of $(\bar{T}, \bar{\mathcal{N}})$ to *S . Since $\varprojlim {}^*T_n \cong \varprojlim T_n$ completion in the maximal ideals gives an isomorphism ${}^*\hat{T} \cong \hat{T}$. By [3, 2.6], [40, 1.3] and [41] there is an isomorphism ${}^*T \cong T \otimes_S {}^*S$ whereby *T is identified with $T \otimes_S {}^*S$. The tower of isomorphisms $\{\sigma_{L'}({}^*\mathcal{N}_n) \cong \mathcal{L}'_n\}$ implies by Lemma 6.1 that there is an isomorphism of deformations $\sigma_{L'}({}^*\mathcal{N}) \cong {}^*\mathcal{L}'$ above $\sigma_{L'}(\bar{\mathcal{N}}) \cong \bar{\mathcal{L}}'$ where ${}^*\mathcal{L}' = {}^*T \otimes_T \mathcal{L}'$.

To apply Artin's Approximation Theorem we define a functor of S -algebras $F : {}_S\mathbf{H} \rightarrow \mathbf{Sets}$ as follows. If \tilde{S} is in ${}_S\mathbf{H}$ let \tilde{T} denote $T \otimes_S \tilde{S}$ and let $\tilde{\mathcal{L}}'$ denote $\tilde{T} \otimes_T \mathcal{L}'$. Then $F(\tilde{S})$ is defined as equivalence classes of pairs of maps of finite \tilde{T} -modules $\tilde{\xi} = (\tilde{\nu} : \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{N}}, \tilde{\iota} : \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{L}}')$ such that $\tilde{\mathcal{N}}$ is \tilde{S} -flat. A map $\tilde{S} \rightarrow \tilde{S}'$ gives a map of pairs by base change. Two pairs, ${}^1\tilde{\xi}$ and ${}^2\tilde{\xi}$, are equivalent if there is an isomorphism ${}^1\tilde{\mathcal{N}} \cong {}^2\tilde{\mathcal{N}}$ commuting with the ${}^j\tilde{\iota}$ and the ${}^j\tilde{\nu}$. We show that F is locally of finite presentation. Suppose $\tilde{S} = \varinjlim {}^iS$ for a filtered injective system of algebras in ${}_S\mathbf{H}$. Put ${}^iT = T \otimes_S {}^iS$. Then $\varinjlim {}^iT \cong \tilde{T}$ by [19, 7.8.3] and [21, 18.6.14]. Given $\tilde{\xi} \in F(\tilde{S})$ as above. Since $\tilde{\mathcal{N}}$ has finite presentation and since the maps $\tilde{\nu}$ and $\tilde{\iota}$ can be represented on the finite presentations, there is a finite iT -module ${}^i\mathcal{N}$ and iT -linear maps ${}^i\nu : {}^i\mathcal{N} \rightarrow \tilde{\mathcal{N}}$ and ${}^i\iota : {}^i\mathcal{N} \rightarrow {}^i\mathcal{L}' = {}^iT \otimes_T \mathcal{L}'$ inducing $\tilde{\xi}$ by base change. We may also assume that ${}^i\mathcal{N}$ is iS -flat. Hence $\varinjlim F({}^iS) \rightarrow F(\tilde{S})$ is surjective, and injectivity is similar. Let ${}^*\xi$ denote the element in $F({}^*S)$ given by ${}^*\mathcal{N} \rightarrow \tilde{\mathcal{N}}$ and the $\hat{\mathbf{D}}$ -hull ${}^*\mathcal{N} \rightarrow {}^*\mathcal{L}'$. By Artin's Approximation Theorem [3, 1.12], [40, 1.3], [41] there exists a $\xi = (\nu : \mathcal{N} \rightarrow \tilde{\mathcal{N}}, \iota : \mathcal{N} \rightarrow \mathcal{L}')$ in $F(S)$ with $\xi_1 = {}^*\xi_1$. In particular $\mathcal{N} \rightarrow \tilde{\mathcal{N}}$ is a deformation. Since $\iota_0 : \mathcal{N}_0 \rightarrow \mathcal{L}'_0$ equals the injective $\mathcal{N} \rightarrow \mathcal{L}'$ Proposition 2.5 applied as in Example 2.6 implies that ι is injective and $\text{coker } \iota$ is S -flat. It follows that ι is the $\hat{\mathbf{D}}$ -hull of \mathcal{N} , i.e. $\sigma_{L'}(\mathcal{N}) = \mathcal{L}'$. The L -case is analogous. \square

Theorem 9.4. *With general assumptions as in Theorem 9.2 consider the map*

$$\sigma_M : \text{Def}_{(A,N)} \longrightarrow \text{Def}_{(A,M)} \text{ of functors } {}_A\mathbf{H}/k \longrightarrow \mathbf{Sets}$$

as in Definition 8.1.

- (i) *If $\text{Hom}_A(L, N) = 0$ then σ_M is injective.*
- (ii) *If $\text{Ext}_A^1(L, N) = 0$ then σ_M is formally smooth.*
- (iii) *If $\text{Ext}_A^1(L, N) = 0$ and $\text{Def}_{(A,N)}$ has a versal element then σ_M is smooth.*

The analogous statements hold for $\sigma_{M'} : \text{Def}_{(A,N)} \rightarrow \text{Def}_{(A,M')}$ with $\text{Ext}_A^1(L', N) = 0$ in (i) and $\text{Ext}_A^2(L', N) = 0$ in (ii) and (iii).

Proof. The proof is analogous to the proof of Theorem 9.2. \square

Fix a field k and a Cohen-Macaulay local algebraic k -algebra A . Let $\Lambda \rightarrow T^\circ$ be obtained by henselisation of a finite type Cohen-Macaulay map $\tilde{\Lambda} \rightarrow \tilde{T}^\circ$ where $\tilde{\Lambda}$ is assumed to be an excellent ring. In particular Λ and T° are excellent rings ([19, 7.8.3], [21, 18.7.6]). Assume $T^\circ/\mathfrak{m}_{T^\circ} \cong k$ and $T^\circ \otimes_A k \cong A$. There is a section ${}_A\mathbf{H} \rightarrow \mathbf{hCM}$ defined by $S \mapsto T^\circ \otimes S$. Let T° denote the fibred subcategory of \mathbf{hCM} . We consider deformations in $\text{mod}_{T^\circ}^{\text{fl}}$ of an object $\xi = (S \rightarrow T^\circ \otimes S, \mathcal{N})$ and obtain the fibred category of deformations $\text{Def}_{\mathcal{N}}^{T^\circ} := (\text{mod}_{T^\circ}^{\text{fl}})_{\text{coca}}/\xi$ over ${}_A\mathbf{H}/S$. The deformation functor $\text{Def}_{\mathcal{N}}^{T^\circ} : {}_A\mathbf{H}/S \rightarrow \mathbf{Sets}$ is defined by the associated groupoid of sets $\overline{\text{Def}}_{\mathcal{N}}^{T^\circ}$. A special case is given by $\Lambda = k$ and $T^\circ = A$.

Corollary 9.5. *With general assumptions as in Theorem 9.2 consider the map $\sigma_{L'} : \text{Def}_N^{T^\circ} \rightarrow \text{Def}_{L'}^{T^\circ}$ of functors ${}_A\mathbf{H}/k \rightarrow \mathbf{Sets}$.*

- (i) *If $\text{Hom}_A(N, M') = 0$ then $\sigma_{L'}$ is injective.*
- (ii) *If $\text{Ext}_A^1(N, M') = 0$ then $\sigma_{L'}$ is formally smooth.*
- (iii) *If $\text{Ext}_A^1(N, M') = 0$ and Def_N^A has a versal element then $\sigma_{L'}$ is smooth.*

The analogous statements hold for $\sigma_L : \text{Def}_N^{T^\circ} \rightarrow \text{Def}_L^{T^\circ}$ with $\text{Ext}_A^1(N, M) = 0$ in (i) and $\text{Ext}_A^2(N, M) = 0$ in (ii) and (iii).

Proof. This is not a formal consequence of Theorem 9.2, but the proof is similar. \square

Corollary 9.6. *With general assumptions as in Theorem 9.2 consider the map $\sigma_M : \text{Def}_N^{T^\circ} \rightarrow \text{Def}_M^{T^\circ}$ of functors ${}_A\mathbf{H}/k \rightarrow \mathbf{Sets}$.*

- (i) *If $\text{Hom}_A(L, N) = 0$ then σ_M is injective.*
- (ii) *If $\text{Ext}_A^1(L, N) = 0$ then σ_M is formally smooth.*
- (iii) *If $\text{Ext}_A^1(L, N) = 0$ and Def_N^A has a versal element then σ_M is smooth.*

The analogous statements hold for $\sigma_{M'} : \text{Def}_N^{T^\circ} \rightarrow \text{Def}_{M'}^{T^\circ}$ with $\text{Ext}_A^1(L', N) = 0$ in (i) and $\text{Ext}_A^2(L', N) = 0$ in (ii) and (iii).

Proposition 9.7. *With general assumptions as in Theorem 9.2:*

- (i) *If $Q' = \text{Hom}_A(\omega_A, L')$ and $Q = \text{Hom}_A(\omega_A, L)$ then Q' and Q have finite projective dimension and $\text{Def}_{(A,L')} \cong \text{Def}_{(A,Q')}$ and $\text{Def}_{(A,L)} \cong \text{Def}_{(A,Q)}$.*
- (ii) *There are natural maps*

$$s : \text{Def}_{(A,M)} \longrightarrow \text{Def}_{(A,M')} \quad \text{and} \quad t : \text{Def}_{(A,L')} \longrightarrow \text{Def}_{(A,L)}$$

commuting with the maps $\sigma_X : \text{Def}_{(A,N)} \rightarrow \text{Def}_{(A,X)}$ for X equal to M and M' , and to L' and L , respectively. If A is a Gorenstein ring, then s is an isomorphism.

The analogous statements also hold for the deformation functors $\text{Def}_X^{T^\circ}$.

Proof. Lemma 6.10 implies (i). There is a short exact sequence $M \rightarrow \omega_A^{\oplus n} \rightarrow M'$ such that the last map is without a common ω_A -summand, corresponding (through dualisation) to a short exact sequence $M^\vee \leftarrow A^{\oplus n} \leftarrow (M')^\vee$ where n is minimal. The map s is the composition $\text{Def}_{(A,M)} \cong \text{Def}_{(A,M^\vee)} \rightarrow \text{Def}_{(A,(M')^\vee)} \cong \text{Def}_{(A,M')}$ where the middle map is obtained by taking the syzygy of the deformation. If A is a Gorenstein ring then $\omega_A \cong A$ and there is an inverse $\text{Def}_{(A,M')} \rightarrow \text{Def}_{(A,M)}$ given by the syzygy map.

Note that the pushout of $M \rightarrow \omega_A^{\oplus n}$ with $M \rightarrow N$ gives $N \rightarrow L'$. Consider the induced short-exact sequence $L \rightarrow \omega_A^{\oplus n} \xrightarrow{\mu} L'$. For a deformation $\lambda' : \mathcal{L}' \rightarrow L'$ there is a lifting of μ to a map $\tilde{\mu} : \omega_h^{\oplus n} \rightarrow \mathcal{L}'$. If \mathcal{L} denotes the kernel of $\tilde{\mu}$ then there is a cocartesian map $\lambda : \mathcal{L} \rightarrow L$ commuting with $\omega_h^{\oplus n} \rightarrow \omega_A^{\oplus n}$. By Lemma 8.2 $\lambda' \mapsto \lambda$ gives a well defined map of deformation functors $t : \text{Def}_{(A,L')} \rightarrow \text{Def}_{(A,L)}$.

Given a deformation (h, \mathcal{N}) in $\text{Def}_{(A,N)}$, let $\mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N}$ and $\mathcal{N} \rightarrow \mathcal{L}' \rightarrow \mathcal{M}'$ be the minimal sequences in Definition 8.1. There is a commutative diagram of deformations with (co)cartesian square

$$(9.7.1) \quad \begin{array}{ccccc} \mathcal{L} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{N} \\ \parallel & & \downarrow & \square & \downarrow \\ \mathcal{L} & \longrightarrow & \omega_h^{\oplus n} & \longrightarrow & \mathcal{L}' \\ & & \downarrow & & \downarrow \\ & & \mathcal{M}' & \equiv & \mathcal{M}' \end{array}$$

where $\omega_h^{\oplus n} \rightarrow \mathcal{L}'$ is given as above. The stated commutativity of maps of deformation functors follows. \square

Corollary 9.8. *Let A be an Cohen-Macaulay local algebraic k -algebra with residue field k . Suppose $\dim A \geq 2$. Then there exists finite A -modules L' and Q' with $\text{inj.dim } L' = \dim A = \text{pdim } Q'$ and universal deformations $\mathcal{L}' \in \text{Def}_{L'}^A(A)$ and $\mathcal{Q}' \in \text{Def}_{Q'}^A(A)$.*

Proof. Let $h = 1 \otimes \text{id} : A \rightarrow A \tilde{\otimes}_k A = T$ and $\mathcal{N} = A$ be the cyclic T -module defined through the multiplication map. Then $T \otimes_A k \cong A$ and $\mathcal{N} \otimes_A k \cong k$ and this gives a deformation $\mathcal{N} \rightarrow k$ of the residue field of A which is universal. If L' is the minimal \hat{D}_A -hull of the residue field k then $\mathcal{L}' = \sigma_{L'}(\mathcal{N}) \in \text{Def}_{L'}^{T^o}(A)$ is universal by Corollary 9.5. If $Q' = \text{Hom}_A(\omega_A, L')$ then $\text{Hom}_T(\omega_T, \mathcal{L}') \in \text{Def}_{Q'}^A(A)$ is universal by Proposition 9.7. \square

Corollary 9.9. *With general assumptions as in Theorem 9.2 put $X = \text{Spec } A$. Let Z be a closed subscheme such that the complement U is contained in the regular locus. Assume $\tilde{N}|_U$ is locally free, $\text{depth}_Z N \geq 2$ and $H_Z^2(\text{Hom}_A(L, N)) = 0$. Then*

$$\sigma_M : \text{Def}_{(A,N)} \longrightarrow \text{Def}_{(A,M)} \quad \text{and} \quad \sigma_M^{T^o} : \text{Def}_N^{T^o} \longrightarrow \text{Def}_M^{T^o} \quad \text{are formally smooth.}$$

Proof. We show that $\text{Ext}_A^1(L, N) = 0$ and apply Theorem 9.4 and Corollary 9.6. By Théorème 1.6 in [22, Exposé VI] there is a cohomological spectral sequence

$$(9.9.1) \quad E_2^{p,q} = \text{Ext}_A^q(L, H_Z^p(N)) \Rightarrow \text{Ext}_Z^{p+q}(X; L, N).$$

Since $H_Z^i(N) = 0$ for $i = 0, 1$ the restriction map $\text{Ext}_A^1(L, N) \rightarrow \text{Ext}_U^1(X; L, N)$ in the long exact sequence is injective. Since U is contained in the regular locus, $\tilde{M}|_U$ and hence $\tilde{L}|_U$ are locally free. It follows that $\text{Ext}_U^1(X; L, N)$ is isomorphic to

$$(9.9.2) \quad \text{Ext}_{\mathcal{O}_U}^1(\tilde{L}|_U, \tilde{N}|_U) \cong H^1(U, \mathcal{H}om_{\mathcal{O}_X}(\tilde{L}, \tilde{N})) \cong H_Z^2(\text{Hom}_A(L, N))$$

which is zero by assumption. \square

Example 9.10. The condition $H_Z^2(\text{Hom}_A(L, N)) = 0$ is implied by $\tilde{N}|_U = 0$ or $\text{depth}_Z(\text{Hom}_A(L, N)) \geq 3$.

The following result extends A. Ishii's [31, 3.2] to deformations of the pair.

Proposition 9.11. *Let k be a field and let A be a Gorenstein local algebraic k -algebra. Suppose $L \rightarrow M \rightarrow N$ is the minimal Cohen-Macaulay approximation of a finite A -module N . If $\text{depth } N = \dim A - 1$ then*

$$\sigma_M : \text{Def}_{(A, N)} \longrightarrow \text{Def}_{(A, M)} \quad \text{and} \quad \sigma_M^{T^\circ} : \text{Def}_N^{T^\circ} \longrightarrow \text{Def}_M^{T^\circ} \quad \text{are smooth.}$$

Proof. By assumption $L \cong A^{\oplus r}$. Assume (h_1, \mathcal{M}_1) in $\text{Def}_{(A, M)}(S_1)$ maps to (h, \mathcal{M}) along the surjection $S_1 \rightarrow S$. Assume σ_M maps (h', \mathcal{N}) in $\text{Def}_{(A, N)}(S)$ to (h, \mathcal{M}) . We can assume that $h' = h$ and that the minimal MCM_h -approximation of \mathcal{N} is $\mathcal{L} \xrightarrow{\rho} \mathcal{M} \rightarrow \mathcal{N}$ where $\mathcal{L} \cong T^{\oplus r}$. Let $\mathcal{L}_1 = T_1^{\oplus r}$ and choose a lifting $\rho_1 : \mathcal{L}_1 \rightarrow \mathcal{M}_1$ of ρ . Put $\mathcal{N}_1 := \text{coker } \rho_1$ with its natural map to \mathcal{N} . Then \mathcal{N}_1 is S_1 -flat ($\rho_1 \otimes S = \rho$) and $\sigma_M(h_1, \mathcal{N}_1) = (h_1, \mathcal{M}_1)$. \square

Remark 9.12. If $\dim A \geq 1$ and an $\text{MCM } A$ -module M has a rank, $r = \text{rk } M$, then there is a short exact sequence $A^r \rightarrow M \rightarrow N$ with N a codimension one Cohen-Macaulay module, cf. [10, 1.4.3]. Hence in the case A is a Gorenstein domain all MCM modules admits MCM_A -approximations by CM modules in codimension one and Proposition 9.11 applies. However, it's not always possible to continue this reduction: If A is a normal Gorenstein complete local ring any $\text{MCM } A$ -module M is the MCM_A -approximation of a codimension 2 Cohen-Macaulay module up to stable isomorphism if and only if A is a unique factorisation domain, see [46, 33].

Let A be a Gorenstein normal domain of dimension 2 and $A^{r-1} \rightarrow M \rightarrow I$ the minimal MCM approximation of a torsion-free rank 1 module I . Let U denote the regular locus in $X = \text{Spec } A$. If $T = A \hat{\otimes}_k S$ for S in ${}_k\mathbf{H}/k$ there is a natural section $A \rightarrow T$. Let U_T denote $U \times_X \text{Spec } T$. Consider the subfunctor $\text{Def}_M^{A, \wedge} \subseteq \text{Def}_M^A$ of deformations \mathcal{M} with trivial induced deformation $\wedge^r \mathcal{M}|_{U_T}$. Note that $H^0(U, \wedge^r M)$ is isomorphic to the MCM module $\bar{I} := H^0(U, I)$. It follows from Proposition 9.11 that the resulting map from the quotient functor $\text{Quot}_{I \subseteq \bar{I}}^I \rightarrow \text{Def}_M^{A, \wedge}$ is also smooth, cf. [31, 3.2]. In particular, if E_A is the fundamental module and $A/\mathfrak{m}_A \cong k$ then $\text{Hom}_{{}_k\mathbf{H}/k}(A, -) \cong \text{Quot}_{\mathfrak{m}_A \subseteq A}^A \cong \text{Def}_{E_A/\mathfrak{m}_A}^A$ gives a versal family for $\text{Def}_{E_A}^{A, \wedge}$ by the MCM approximation in Corollary 7.3, see [31, 3.4].

Example 9.13. Assume $A/\mathfrak{m}_A \cong k$ and let M denote the minimal MCM approximation of k . It's given as $M \cong \text{Hom}_A(\text{Syz}_d^A(k^\vee), \omega_A)$ where $d = \dim A$, cf. Remark 5.6. One has $k^\vee = \text{Ext}_A^d(k, \omega_A) \cong k$. We apply $\text{Hom}_A(-, \omega_A)$ to the short exact sequence $\text{Syz}^A(\mathfrak{m}_A) \rightarrow A^{\oplus \beta_1} \xrightarrow{(\underline{x})} \mathfrak{m}_A$. Assume $\dim A = 2$. Since $\text{Ext}_A^1(\mathfrak{m}_A, \omega_A) \cong k$ we obtain the MCM approximation of k :

$$(9.13.1) \quad 0 \rightarrow \omega_A \xrightarrow{(\underline{x})^\dagger} \omega_A^{\oplus \beta_1} \rightarrow M \rightarrow k \rightarrow 0$$

In particular $\text{rk}(M) = \beta_1 - 1$ and $\mu(M) = t(A) \cdot \beta_1 + 1$ where $t(A)$ is the Cohen-Macaulay type of A . In the case $A = A(m) = k[u^m, u^{m-1}v, \dots, v^m]^h$, the vertex of the cone over the rational normal curve of degree m , which has the indecomposable MCM modules $M_i = (u^i, u^{i-1}v, \dots, v^i)$ for $i = 0, \dots, m-1$, one finds that $M = M_{m-1}^{\oplus m}$. We have

$$(9.13.2) \quad \dim_k \text{Def}_M^A(k[\varepsilon]) = \dim_k \text{Ext}_A^1(M, M) = (m-1) \cdot m^2$$

while $\dim_k \text{Def}_k^A(k[\varepsilon]) = m + 1$. Even in the Gorenstein case ($m = 2$) the tangent map isn't surjective and so Proposition 9.11 cannot in general be extended to $\text{depth } N = \dim A - 2$. For a detailed description of the strata defined by Ishii in

[31] of the reduced versal deformation space of M , see [23]. Applying $\text{Hom}_A(k, -)$ to $\mathfrak{m}_A \rightarrow A \rightarrow k$ gives an exact sequence

$$(9.13.3) \quad 0 \rightarrow \text{Ext}_A^1(k, k) \rightarrow \text{Def}_{\mathfrak{m}_A}^A(k[\varepsilon]) \rightarrow k^{t(A)} \rightarrow \text{Ext}_A^2(k, k)$$

since $\text{Ext}_A^1(\mathfrak{m}_A, \mathfrak{m}_A) \cong \text{Ext}_A^2(k, \mathfrak{m}_A)$ and $\dim A = 2$. In the case $A = A(m)$ the fundamental module E_A (7.1.1) is isomorphic to $M_{m-1}^{\oplus 2}$ with $\dim_k \text{Def}_{E_A}^A(k[\varepsilon]) = 4(m-1)$. The conclusion in Proposition 9.11 cannot hold in the non-Gorenstein case $m > 2$.

10. EXISTENCE OF VERSAL ELEMENTS

We prove existence of a versal element for a pair (algebra, module) with isolated singularity. The following lemma is used in the proof.

Lemma 10.1. *Let $h^{\text{ft}} : S \rightarrow T^{\text{ft}}$ be a finite type homomorphism of noetherian rings. Let $\Gamma^{\text{ft}} = T^{\text{ft}} \oplus N$ be the graded S -algebra with a finite T^{ft} -module N in degree 1 and let $I = {}^0I \oplus {}^1I$ be a graded Γ^{ft} -module with iI in degree i . Let T denote the henselisation of T^{ft} in a maximal ideal \mathfrak{m} and put $\Gamma = T \otimes \Gamma^{\text{ft}}$.*

(a) *There are natural isomorphisms of graded André-Quillen cohomology*

$${}_0H^i(S, \Gamma, T \otimes I) \cong {}_0H^i(S, \Gamma^{\text{ft}}, I) \otimes_{T^{\text{ft}}} T \quad \text{for all } i.$$

Suppose in addition that h^{ft} is flat, I is finite as T^{ft} -module, S is local henselian and $S/\mathfrak{m}_S \cong T^{\text{ft}}/\mathfrak{m} \cong k$. Let $k \rightarrow A^{\text{ft}}$ denote the central fibre of h^{ft} . Put $\mathfrak{m}_0 = \mathfrak{m}_A^{\text{ft}}$ and $N_0 = N \otimes_S k$. Assume $V = \text{Spec } A^{\text{ft}} \setminus \{\mathfrak{m}_0\}$ is smooth over k and N_0 restricted to V is locally free.

(b) *For all $i > 0$ the graded André-Quillen cohomology ${}_0H^i(S, \Gamma, T \otimes I)$ is finite as S -module and there is a natural $T_{\mathfrak{m}}^{\text{ft}}$ -isomorphism*

$${}_0H^i(S, \Gamma, T \otimes I) \cong {}_0H^i(S, \Gamma^{\text{ft}}, I)_{\mathfrak{m}}.$$

Proof. (a) Note that there are natural maps ${}_0H^i(S, \Gamma, T \otimes I) \rightarrow {}_0H^i(S, \Gamma^{\text{ft}}, T \otimes I)$ and ${}_0H^i(S, \Gamma^{\text{ft}}, I) \otimes_{T^{\text{ft}}} T \rightarrow {}_0H^i(S, \Gamma^{\text{ft}}, T \otimes I)$. We consider the natural long-exact sequence in Proposition 8.8 and prove that the analogous maps for the ungraded André-Quillen and Ext cohomology are isomorphisms. The cotangent complex is trivial for Zariski localisation and by the transitivity sequence $H_i(T^{\text{ft}}, T, -) \rightarrow H_i(T_{\mathfrak{m}}^{\text{ft}}, T, -)$ is an isomorphism for all i . André-Quillen homology commutes with direct limits in the second argument [2, III 35] and the cotangent complex is trivial for étale extensions [30, III 3.1.1]. Hence $H_i(T^{\text{ft}}, T, T) = 0$ for all i . By [2, III 21]

$$(10.1.1) \quad H^i(T^{\text{ft}}, T, T \otimes {}^0I) \cong \text{Hom}_T(H_i(T^{\text{ft}}, T, T), T \otimes {}^0I) = 0 \quad \text{for all } i.$$

From the transitivity sequence $H^i(S, T, T \otimes {}^0I) \cong H^i(S, T^{\text{ft}}, T \otimes {}^0I)$ for all i . Moreover $T^{\text{ft}} \rightarrow T$ is flat and T^{ft} is of finite type over the noetherian S and we obtain

$$(10.1.2) \quad H^i(S, T^{\text{ft}}, {}^0I) \otimes_{T^{\text{ft}}} T \cong H^i(S, T^{\text{ft}}, T \otimes {}^0I) \cong H^i(S, T, T \otimes {}^0I)$$

for all i [2, IV 58]. Since T is T^{ft} -flat and N finite

$$(10.1.3) \quad \text{Ext}_{T^{\text{ft}}}^i(N, {}^1I) \otimes_{T^{\text{ft}}} T \cong \text{Ext}_{T^{\text{ft}}}^i(N, T \otimes {}^1I) \cong \text{Ext}_T^i(T \otimes N, T \otimes {}^1I).$$

(b) The non-smooth locus of h^{ft} is closed, i.e. defined by an ideal $J \subseteq T^{\text{ft}}$. Smooth is equivalent to flat with smooth fibres [21, 17.5.1]. Hence $J_0 = JA^{\text{ft}}$ defines the non-smooth locus of $k \rightarrow A^{\text{ft}}$ and J_0 is \mathfrak{m}_0 -primary. Put $\bar{T}^{\text{ft}} = T^{\text{ft}}/J$. Since A^{ft}/J_0 has finite length, $S \rightarrow \bar{T}^{\text{ft}}$ is quasi-finite at \mathfrak{m} [18, Err 20] by Chevalley's upper semi-continuity theorem [20, 13.1.3] and openness of $\text{Spec } T^{\text{ft}} \rightarrow \text{Spec } S$ [19, 2.4.6]. Since S is henselian it follows that there is a ring T' such that \bar{T}^{ft} is isomorphic to $\bar{T}_{\mathfrak{m}}^{\text{ft}} \amalg T'$ where $\bar{T}_{\mathfrak{m}}^{\text{ft}}$ is finite as S -module [21, 18.5.11]. Hence there is a Zariski neighborhood U of \mathfrak{m} in $\text{Spec } T^{\text{ft}}$ such that non-smooth locus $U \cap V(J)$ is finite over

S and the support of $H^i = H^i(S, T^{\text{ft}}, {}^1I)$ in U is contained in $U \cap V(J)$ for all $i > 0$ by [30, III 3.1.2]. Since $H_{\mathfrak{m}}^i$ equals H^i restricted to U , it follows that $H_{\mathfrak{m}}^i$ is finite as S -module. With the isomorphism in (a) we get

$$(10.1.4) \quad H^i(S, T, T \otimes {}^0I) \cong H^i(S, T^{\text{ft}}, {}^0I)_{\mathfrak{m}} \otimes_{T^{\text{ft}}} T \cong H^i(S, T^{\text{ft}}, {}^0I)_{\mathfrak{m}}.$$

The locus where N isn't locally free is closed, i.e. defined by an ideal $J' \subseteq T^{\text{ft}}$. Locally free is equivalent to flat and locally free fibres. Hence $J'_0 = J' A^{\text{ft}}$ defines the singular locus of N_0 and J'_0 is \mathfrak{m}_0 -primary. As for the André-Quillen cohomology we get a Zariski neighborhood U' of \mathfrak{m} such that $E^i = \text{Ext}_{T^{\text{ft}}}^i(N, {}^1I)$ restricted to U' equals $E_{\mathfrak{m}}^i$ and is finite as S -module for all $i > 0$. With (a) we get

$$(10.1.5) \quad \text{Ext}_T^i(T \otimes N, T \otimes {}^1I) \cong \text{Ext}_{T^{\text{ft}}}^i(N, {}^1I) \otimes_{T^{\text{ft}}} T \cong \text{Ext}_{T^{\text{ft}}}^i(N, {}^1I)_{\mathfrak{m}}.$$

We conclude by the long-exact sequence in Proposition 8.8. \square

Let k be a field, A an algebraic k -algebra with $A/\mathfrak{m}_A \cong k$ and N a finite A -module. Without any Cohen-Macaulay condition on A we define a deformation $(h : S \rightarrow T, \mathcal{N})$ of the pair (A, N) to an S in ${}_A\mathbf{H}/k$ as before and obtain the deformation functor $\text{Def}_{(A, N)} : {}_A\mathbf{H}/k \rightarrow \mathbf{Sets}$ as equivalence classes of deformations of pairs.

We say that A is an *isolated singularity over k* if there is a finite type k -algebra A^{ft} with a maximal ideal \mathfrak{m}_0 such that the henselisation $(A^{\text{ft}})_{\mathfrak{m}_0}^{\text{h}}$ is isomorphic to A and which is smooth over k at all points in $\text{Spec } A^{\text{ft}} \setminus \{\mathfrak{m}_0\}$. We say that the pair (A, N) is an *isolated singularity over k* if A is an isolated singularity over k and if $N_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module for all prime ideals $\mathfrak{p} \neq \mathfrak{m}_A$.

The following theorem is a consequence of results of R. Elkik and an argument of H. von Essen.

Theorem 10.2. *Let (A, N) be an isolated singularity over the field k with A equidimensional. Then $\text{Def}_{(A, N)} : {}_A\mathbf{H}/k \rightarrow \mathbf{Sets}$ has a versal element.*

Proof. We apply [4, 3.2] with the extension to arbitrary excellent coefficients given by [13, 1.5] to show the existence of a formally versal element for $\text{Def}_{(A, N)}$. By the finiteness conditions it follows that $\text{Def}_{(A, N)}$ is locally of finite presentation. The condition (S1) holds in general by Proposition 8.5. Let $(h : S \rightarrow T, \mathcal{N})$ be a deformation to S . Let $(T^{\text{ft}}, \mathfrak{m})$ be an S -flat finite type representative for T such that $A^{\text{ft}} = T^{\text{ft}} \otimes_S k$ has a single non-smooth closed point $\mathfrak{m}_0 = \mathfrak{m} A^{\text{ft}}$ and let \mathcal{N}^{ft} be an S -flat finite T^{ft} -module representing \mathcal{N} . The singular locus of $N^{\text{ft}} = \mathcal{N}^{\text{ft}} \otimes_S k$ is closed and equal to $V(J_0)$ for some ideal $J_0 \subseteq A^{\text{ft}}$. But \mathfrak{m}_0 is isolated in $V(J_0)$ so (possibly after inverting some element in T^{ft}) we may assume that N^{ft} is locally free away from \mathfrak{m}_0 . Let I be a finite S -module. By Lemma 10.1 ${}_0H^1(S, \Gamma, \Gamma \otimes_S I)$ with $\Gamma = T \oplus \mathcal{N}$ is a finite S -module, so by Proposition 8.5 condition (S2) holds.

For effectivity, there is a deformation functor $\text{Def}_{(A^{\text{ft}}, N^{\text{ft}})} : {}_A\mathbf{H}/k \rightarrow \mathbf{Sets}$ of base change maps of pairs $(S \rightarrow T^{\text{ft}}, \mathcal{N}^{\text{ft}}) \rightarrow (k \rightarrow A^{\text{ft}}, N^{\text{ft}})$ where T^{ft} is a flat S -algebra of finite type and \mathcal{N}^{ft} is an S -flat finite T^{ft} -module. Base change is given by the standard tensor product. Similarly there is a $\text{Def}_{A^{\text{ft}}}$. Restricted to ${}_A\mathbf{A}/k$ $\text{Def}_{(A^{\text{ft}}, N^{\text{ft}})}$ satisfies (S1) and (S2). Hence there exists a formally versal formal element $\{(T_n^{\text{ft}}, \mathcal{N}_n^{\text{ft}})\}$ in $\varprojlim \text{Def}_{(A^{\text{ft}}, N^{\text{ft}})}(S_n)$ where $S_n = S/\mathfrak{m}_S^{n+1}$ for some $S = \hat{S}$ in ${}_A\mathbf{H}/k$. By [16, Théorème 7, p. 595] (cf. [5, II 5.1]) there exists an element $S \rightarrow T^{\text{ft}}$ in $\text{Def}_{A^{\text{ft}}}(S)$ which induces $\{T_n^{\text{ft}}\}$. Let T' be the henselisation of T^{ft} in the maximal ideal $\mathfrak{m} = (T^{\text{ft}} \rightarrow A)^{-1}(\mathfrak{m}_A)$. Then $S \rightarrow T'$ is a deformation of A . Let T^* be the completion of T' at the ideal $\mathfrak{n} = \mathfrak{m}_S T'$ and let $\mathcal{N}^* = \varprojlim \mathcal{N}_n$. Then \mathcal{N}^* is an S -flat finite T^* -module. Let $J^* \subseteq T^*$ denote the ideal $I(\varphi)$ where φ is a minimal presentation of \mathcal{N}^* . Then J^* defines the locus $V(J^*)$ where \mathcal{N}^* is not locally free.

Let $J = \ker(T' \rightarrow T^*/J^*)$. Since T^*/J^* is finite as S -module, $T'/J \cong T^*/J^*$. The proof of [44, 2.3] works in this situation too (there is a typo in line 5: it should be a direct sum, not a tensor product) and shows that the completion of T' in the ideal $\mathfrak{a} = J \cap \mathfrak{m}_S T'$ equals T^* . Since \mathcal{N}^* is locally free on the complement of $V(\mathfrak{a}T^*)$, the conditions in [16, Théorème 3] hold. From this result we obtain a finite T' -module \mathcal{N} inducing \mathcal{N}^* . In particular \mathcal{N} is S -flat.

We claim that the henselisation map $\mathrm{Def}_{(A^{\mathrm{ft}}, N^{\mathrm{ft}})} \rightarrow \mathrm{Def}_{(A, N)}$ is formally smooth. It follows that the element (T', \mathcal{N}) in $\mathrm{Def}_{(A, N)}(S)$ is formally versal. For the claim, put $\Gamma^{\mathrm{ft}} = A^{\mathrm{ft}} \oplus N^{\mathrm{ft}}$ and $\Gamma = A \oplus N$ and let $\pi : S_1 \rightarrow S_0 = S_1/I$ be a small surjection in ${}_A\mathbf{A}/k$. The obstruction $\mathrm{ob}(\pi, \Gamma_0^{\mathrm{ft}}) \in {}_0H^2(k, \Gamma^{\mathrm{ft}}, \Gamma^{\mathrm{ft}}) \otimes_k I$ for lifting a deformation Γ_0^{ft} of Γ^{ft} along π maps to the corresponding obstruction $\mathrm{ob}(\pi, \Gamma_0) \in {}_0H^2(k, \Gamma, \Gamma) \otimes_k I$. The isomorphisms $H^i(S, T, T) \cong H^i(S, T^{\mathrm{ft}}, T^{\mathrm{ft}}) \otimes_{T^{\mathrm{ft}}} T$ for all i implies isomorphisms ${}_0H^i(k, \Gamma^{\mathrm{ft}}, \Gamma^{\mathrm{ft}}) \cong {}_0H^i(k, \Gamma, \Gamma)$ for $i = 1, 2$ as in the beginning of the proof. Smoothness follows by the standard obstruction argument. By [4, 3.2] and [13] there is an algebraic k -algebra R and a formally versal element (T, \mathcal{N}) in $\mathrm{Def}_{(A, N)}(R)$.

Finally we apply [4, 3.3] (for general excellent coefficients) to conclude that the formally versal element (T, \mathcal{N}) is versal. We already have (S1) and (S2). To check [4, 3.3(iii)], let S be algebraic in ${}_A\mathbf{A}/k$, I an ideal in S and put $S^* = \varprojlim S_n$ where $S_n = S/I^{n+1}$. Let ${}^i\xi = ({}^iT, {}^i\mathcal{N})$ for $i = 1, 2$ be two elements in $\mathrm{Def}_{(A, N)}(S^*)$ and $\{\theta_n : {}^1\xi_n \cong {}^2\xi_n\}$ be a tower of isomorphisms between the S_n -truncations. There are finite type representatives ${}^i\xi^{\mathrm{ft}} = ({}^iT^{\mathrm{ft}}, {}^i\mathcal{N}^{\mathrm{ft}})$ of the ${}^i\xi$. By the cohomology argument above one obtains by induction a tower of isomorphisms $\{\theta_n^{\mathrm{ft}} : {}^1\xi_n^{\mathrm{ft}} \cong {}^2\xi_n^{\mathrm{ft}}\}$ inducing $\{\theta_n\}$. Since $\varprojlim \hat{S}/I^{n+1} \hat{S} \cong S^*$ where \hat{S} is the completion of S in the maximal ideal, we can apply [16, Lemme p. 600] to conclude that the henselisations of the ${}^iT^{\mathrm{ft}}$ in ${}^iT^{\mathrm{ft}}I$ are isomorphic by an isomorphism lifting $\theta_0 : {}^1T_0 \cong {}^2T_0$. Further henselisation in the maximal ideals gives an isomorphism of deformations ${}^1T \cong {}^2T$. By Lemma 6.1 the isomorphism is extended to an isomorphism of the pairs $\psi : {}^1\xi \cong {}^2\xi$ which lifts θ_0 . By [44, 1.3] condition [4, 3.3(ii)] is unnecessary and we conclude that (T, \mathcal{N}) is versal. \square

Remark 10.3. Let A be an Cohen-Macaulay local algebraic k -algebra and N a finite A -module. We say that N has an *isolated singularity* if $N_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module for all prime ideals $\mathfrak{p} \neq \mathfrak{m}_A$. In that case a similar, but easier argument gives that $\mathrm{Def}_N^{T_0}$ has a versal element. This is the result [44, 2.4] of von Essen, but for a slightly different fibred category of deformations where henselisation is taken along the closed fibre. However it implies the result in our case, essentially by henselisation at \mathfrak{m}_0 . Corollary 10.4 and 10.5 have obvious analogs for $\mathrm{Def}_N^{T_0}$ in this case.

Corollary 10.4. *Suppose A is an isolated Cohen-Macaulay singularity over the field k and N is a finite length A -module. Let $L \rightarrow M \rightarrow N$ and $N \rightarrow L' \rightarrow M'$ be the minimal MCM_A -approximation and \hat{D}_A -hull of N respectively. Then:*

- (i) $\mathrm{Def}_{(A, N)}$ has a versal element.
- (ii) If $\dim A \geq 2$ and Q' denotes $\mathrm{Hom}_A(\omega_A, L')$ then

$$\mathrm{Def}_{(A, N)} \cong \mathrm{Def}_{(A, L')} \cong \mathrm{Def}_{(A, Q')}.$$

- (iii) If $\dim A \geq 3$ and Q denotes $\mathrm{Hom}_A(\omega_A, L)$ then

$$\mathrm{Def}_{(A, N)} \cong \mathrm{Def}_{(A, L)} \cong \mathrm{Def}_{(A, Q)}.$$

Proof. This is Theorem 10.2, Theorem 9.2 and Proposition 9.7. \square

Corollary 10.5. *Suppose A is a local algebraic k -algebra which is a Gorenstein normal domain with $\dim A = 2$ and N is a finite torsion-free A -module. Let $L \rightarrow M \rightarrow N$ be the minimal MCM_A -approximation of N . Assume k is perfect. Then $\text{Def}_{(A,N)}$ and $\text{Def}_{(A,M)}$ both have versal elements and the map $\sigma_M : \text{Def}_{(A,N)} \rightarrow \text{Def}_{(A,M)}$ is smooth.*

Proof. Since A is a domain N torsion-free implies that N is a first syzygy. It follows that $N_{\mathfrak{p}}$ is a $\text{MCM}_{A_{\mathfrak{p}}}$ -module for all primes $\mathfrak{p} \neq \mathfrak{m}_A$ and since A is 2-dimensional and normal $N_{\mathfrak{p}}$ is free. As k is perfect it follows from [19, 6.7.7 and 6.8.6] that (A, N) and (A, M) are isolated singularities and hence Theorem 10.2 applies. Since $\text{depth } N \geq 1$, L is projective and by Theorem 9.4 (iii) σ_M is smooth. \square

11. DEFORMING MAXIMAL COHEN-MACAULAY APPROXIMATIONS OF COHEN-MACAULAY MODULES

Let $h : S \rightarrow T$ be a homomorphism of local noetherian rings. An h -sequence (or just an h -regular element if $n = 1$) is a sequence $J = (f_1, \dots, f_n)$ in T such that the image \bar{J} in $A = T \otimes_S S/\mathfrak{m}_S$ is an A -sequence. Applying the Koszul complex $K(J)$ as in Example 2.6 one sees that an h -sequence is a transversally T -regular sequence relative to S as defined in [21, 19.2.1]. In particular; J is an h -sequence if and only if J is a T -sequence and T/J is S -flat.

Theorem 11.1. *Let $q : A \rightarrow T^\circ$ denote the henselisation of a finite type Cohen-Macaulay map with $T^\circ/\mathfrak{m}_{T^\circ} = k$ and $T^\circ \otimes_A k = A$. Suppose $J = (f_1, \dots, f_n)$ is a q -sequence. Put $\bar{T}^\circ = T^\circ/J$, $B = \bar{T}^\circ \otimes_A k$ and let \bar{J} be the image of J in A . Let N be a maximal Cohen-Macaulay B -module and $L \rightarrow M \rightarrow N$ the minimal MCM_A -approximation of N . If $\text{ob}(A/\bar{J}^2 \rightarrow B, N) = 0$, then the composition of maps*

$$\text{Def}_N^{\bar{T}^\circ} \longrightarrow \text{Def}_N^{T^\circ} \longrightarrow \text{Def}_M^{T^\circ}$$

*of functors from ${}_A\mathbf{H}/k$ to **Sets** is injective.*

Example 11.2. The existence of a splitting $B \rightarrow A/\bar{J}^2$ implies that $\text{ob}(A/\bar{J}^2, N) = 0$ for all B -modules N since $A/\bar{J}^2 \otimes_B N$ gives a lifting of N to A/\bar{J}^2 .

Let \mathbf{C} be a category. Then $\text{Arr } \mathbf{C}$ denotes the category with objects being arrows in \mathbf{C} and arrows being commutative diagrams of arrows in \mathbf{C} . An endo-functor F on \mathbf{C} induces an endo-functor $\text{Arr } F$ on $\text{Arr } \mathbf{C}$. Let B be a noetherian local ring and \mathbf{P}_B the additive subcategory of projective modules in mod_B . Let $\underline{\text{Hom}}_B(N, M)$ denote the homomorphisms from N to M in the quotient category $\underline{\text{mod}}_B = \text{mod}_B/\mathbf{P}_B$ i.e. B -homomorphisms modulo the ones factoring through an object in \mathbf{P}_B . For each N in mod_B we fix a minimal B -free resolution and use it to define the syzygy modules of N . Then the association $N \mapsto \text{Syz}_i^B N$ induces an endo-functor on $\underline{\text{mod}}_B$ for each i , considered by A. Heller [26]. Define $\underline{\text{Ext}}_B^i(N, M)$ as $\underline{\text{Hom}}_B(\text{Syz}_i^B N, M)$ which turns out to be isomorphic to $\text{Ext}_B^i(N, M)$ for all $i > 0$.

Lemma 11.3. *Let A be a noetherian local ring and $I = (f_1, \dots, f_n)$ a regular sequence. Put $B = A/I$ and suppose N, N_1 and N_2 are finite B -modules. Let \bar{M}_i denote $B \otimes \text{Syz}_n^A N_i$.*

- (i) *There is an inclusion $u_N : N \rightarrow \bar{M}_N$ of B -modules with $\bar{M}_N \cong B \otimes \text{Syz}_n^A N$ which induces a functor $u : \underline{\text{mod}}_B \rightarrow \text{Arr } \underline{\text{mod}}_B$.*
- (ii) *The functor u commutes with the B -syzygy functor:*

$$\text{Arr } \text{Syz}_i^B(u_N) = u_{\text{Syz}_i^B N}$$

- (iii) The endo-functor $B \otimes \text{Syz}_n^A(-)$ induces a natural map $\underline{\text{Ext}}_B^i(N_1, N_2) \rightarrow \underline{\text{Ext}}_B^i(\bar{M}_1, \bar{M}_2)$ which makes the following diagram commutative:

$$\begin{array}{ccc} \underline{\text{Ext}}_B^i(N_1, N_2) & \xrightarrow{\quad} & \underline{\text{Ext}}_B^i(\bar{M}_1, \bar{M}_2) \\ & \searrow (u_{N_2})^* \quad \swarrow (u_{N_1})^* & \\ & \underline{\text{Ext}}_B^i(N_1, \bar{M}_2) & \end{array}$$

- (iv) The inclusion $u_N : N \hookrightarrow B \otimes \text{Syz}_n^A N$ splits $\iff \text{ob}(A/J^2 \rightarrow B, N) = 0$.

Remark 11.4. Lemma 11.3 (iv) strengthens [8, 3.6] (in the commutative case).

Proof. (i): Suppose $F_* \rightarrow N$ is a minimal A -free resolution of N . Tensoring the short exact sequence $\text{Syz}_n^A N \xrightarrow{j} F_{n-1} \rightarrow \text{Syz}_{n-1}^A N$ with B gives the exact sequence

$$(11.4.1) \quad 0 \rightarrow \text{Tor}_1^A(B, \text{Syz}_{n-1}^A N) \rightarrow B \otimes \text{Syz}_n^A N \rightarrow \bar{F}_{n-1} \rightarrow B \otimes \text{Syz}_{n-1}^A N \rightarrow 0.$$

We have $\text{Tor}_1^A(B, \text{Syz}_{n-1}^A N) \cong \text{Tor}_n^A(B, N) \cong N$. Let u_N be the inclusion $N \cong \ker(B \otimes_A j) \rightarrow B \otimes \text{Syz}_n^A N$. Then $N \mapsto u_N$ gives a functor of quotient categories.

(ii): Let $p : Q \rightarrow N$ be the minimal B -free cover and $P_* \rightarrow \text{Syz}^B N$ the minimal A -free resolution of the B -syzygy $\ker(p)$. Then there is an A -free resolution $H_* \rightarrow Q$ which is an extension of F_* by P_* . Since $\text{Syz}_n^A B \cong A$, tensoring the short exact sequence of resolutions by B we obtain the commutative diagram with exact rows:

$$(11.4.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Syz}^B N & \longrightarrow & Q & \longrightarrow & N \longrightarrow 0 \\ & & \downarrow u_{\text{Syz}^B N} & & \downarrow & & \downarrow u_N \\ 0 & \longrightarrow & B \otimes \text{Syz}_n^A(\text{Syz}^B N) & \longrightarrow & B^r \oplus Q & \longrightarrow & B \otimes \text{Syz}_n^A N \longrightarrow 0 \end{array}$$

which proves the claim.

(iii): By (ii) it is enough to prove this for $i = 0$. The case $i = 0$ follows from the functoriality in (i).

(iv, \Leftarrow): For the case $n = 1$ see the proof of [8, 3.2]. Assume $n \geq 2$. We follow the proof of [8, 3.6] closely. Let $A_1 = A/(f_1)$. Then $F_*^{(1)} = A_1 \otimes F_{* \geq 1}[1]$ gives a minimal A_1 -free resolution of $A_1 \otimes \text{Syz}^A N$. We have $\text{ob}(A/J^2 \rightarrow B, N) = 0 \Rightarrow \text{ob}(A/(f_1)^2 \rightarrow A_1, N) = 0$ and hence N is a direct summand of $A_1 \otimes \text{Syz}^A N$. Let $G_* \rightarrow N$ be a minimal A_1 -free resolution of N . Then G_* is a direct summand of $F_*^{(1)}$ and hence $\text{Syz}_{n-1}^{A_1} N$ is a direct summand of $\text{Syz}_{n-1}^{A_1}(A_1 \otimes \text{Syz}^A N) = A_1 \otimes \text{Syz}_n^A N$. Tensoring this situation with B gives a commutative diagram:

$$(11.4.3) \quad \begin{array}{ccccccc} N & \xrightarrow{u} & B \otimes \text{Syz}_n^A N & \xrightarrow{\bar{j}} & \bar{F}_{n-1} & \longrightarrow & \cdots \longrightarrow \bar{F}_1 \longrightarrow \bar{F}_0 \longrightarrow N \\ \parallel & & \updownarrow & & \updownarrow & & \updownarrow \\ N & \xrightarrow{u_1} & B \otimes \text{Syz}_{n-1}^{A_1} N & \xrightarrow{\bar{j}_1} & \bar{G}_{n-2} & \longrightarrow & \cdots \longrightarrow \bar{G}_0 \longrightarrow N \end{array}$$

Since $\text{ob}(A/J^2 \rightarrow B, N) = 0 \Rightarrow \text{ob}(A_1/(f_2, \dots, f_n)^2 \rightarrow B, N) = 0$ the map u_1 splits by induction on n . So u splits. The other direction follows from [8, 3.6]. \square

Proposition 11.5. Let $h : S \rightarrow T$ be a local Cohen-Macaulay map, $J = (f_1, \dots, f_n)$ an h -sequence, $\bar{h} : S \rightarrow \bar{T} = T/J$ the local Cohen-Macaulay map induced from h , and (\bar{h}, \mathcal{N}) an object in $\text{MCM}_{\bar{h}}$. Let $\xi : \mathcal{L} \rightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{N}$ be the minimal MCM_h -approximation of \mathcal{N} . Then tensoring ξ by \bar{T} gives a 4-term exact sequence

$$0 \rightarrow \mathcal{N} \otimes J/J^2 \longrightarrow \bar{\mathcal{L}} \longrightarrow \bar{\mathcal{M}} \xrightarrow{\bar{\pi}} \mathcal{N} \rightarrow 0$$

which represents the obstruction class $\text{ob}(T/J^2 \rightarrow \bar{T}, \mathcal{N}) \in \text{Ext}_{\bar{T}}^2(\mathcal{N}, \mathcal{N} \otimes J/J^2)$.

Moreover; $\text{ob}(T/J^2 \rightarrow \bar{T}, \mathcal{N}) = 0 \iff \text{ob}(T/J^2 \rightarrow \bar{T}, \mathcal{N}^\vee) = 0 \iff \bar{\pi}$ splits where $\mathcal{N}^\vee = \text{Ext}_T^n(\mathcal{N}, \omega_h)$.

Proof. By Proposition 2.5 $\mathrm{Tor}_i^T(\bar{T}, \mathcal{M}) = \mathrm{H}_i(K(\underline{f}) \otimes \mathcal{M}) = 0$ for $i > 0$. There is a map from the defining short exact sequence $\mathrm{Syz}^T \mathcal{N} \rightarrow F_0 \rightarrow \mathcal{N}$ to ξ lifting $\mathrm{id}_{\mathcal{N}}$. Tensoring with \bar{T} gives a map of 4-term exact sequences with outer terms canonically identified. Hence they represent the same element $\mathrm{ob}(T/J^2 \rightarrow \bar{T}, \mathcal{N})$ in $\mathrm{Ext}_{\bar{T}}^2(\mathcal{N}, \mathcal{N} \otimes J/J^2)$.

By the argument in Remark 5.6 we can assume that ξ is given as $\mathrm{im}(d_n^\vee) \rightarrow (\mathrm{Syz}_n^T \mathcal{N}^\vee)^\vee \rightarrow \mathcal{N}^{\vee\vee}$ where (F_*, d_*) is a minimal T -free resolution of \mathcal{N}^\vee . By Lemma 11.3 $\mathrm{ob}(T/J^2 \rightarrow \bar{T}, \mathcal{N}^\vee) = 0$ if and only if $u : \mathcal{N}^\vee \rightarrow \bar{T} \otimes \mathrm{Syz}_n^T \mathcal{N}^\vee$ splits. But applying $\mathrm{Hom}_{\bar{T}}(-, \omega_{\bar{h}})$ to u gives $\bar{\pi}$ since $\mathcal{N} \cong \mathrm{Ext}_T^n(\mathcal{N}^\vee, \omega_h) \cong \mathrm{Hom}_{\bar{T}}(\mathcal{N}^\vee, \omega_{\bar{h}})$. \square

Remark 11.6. In the absolute Gorenstein case with $n = 1$ this is given in [8, 4.5].

Proof of Theorem 11.1. Given S in ${}_A \mathbf{H}/k$ and let $h : S \rightarrow T$ and $\bar{h} : S \rightarrow \bar{T} = T/JT$ be the induced hCM maps. Let ${}^i\mathcal{N}$ be deformations of N to \bar{h} for $i = 1, 2$ and assume that the minimal MCM_h -approximation modules ${}^i\mathcal{M}$ of ${}^i\mathcal{N}$ are isomorphic as deformations of M . We proceed as in the proof of Theorem 9.2 (i) with $S_n = S/\mathfrak{m}_S^{n+1}$, $\mathcal{N}_n = \mathcal{N} \otimes_S S_n$ etc., construct a tower of isomorphisms $\{\varphi_n : {}^1\mathcal{N} \cong {}^2\mathcal{N}\}$, and conclude by Lemma 6.1 that ${}^1\mathcal{N}$ and ${}^2\mathcal{N}$ are isomorphic as deformations of N . For the induction step we use that the map of torsor actions along $\mathrm{Def}_{\mathcal{N}_n}^{\bar{T}}(S_{n+1}) \rightarrow \mathrm{Def}_{\mathcal{M}_n}^T(S_{n+1})$ is induced by a natural map $p : \mathrm{Ext}_B^1(N, N) \rightarrow \mathrm{Ext}_A^1(M, M)$ which is injective. The map p is given as follows.

Let $\pi : M \rightarrow N$ denote the MCM_A -approximation and $\bar{\pi} : \bar{M} \rightarrow N$ be the B -quotient. Then $\bar{\pi}$ splits by Proposition 11.5. Hence $\bar{\pi}^* : \mathrm{Ext}_B^1(N, N) \hookrightarrow \mathrm{Ext}_B^1(\bar{M}, N)$ splits. Since J is an M -regular sequence, $\mathrm{Ext}_B^1(\bar{M}, N) \cong \mathrm{Ext}_A^1(M, N)$. Since $\mathrm{Ext}_A^i(M, L) = 0$ for all $i > 0$, $\pi_* : \mathrm{Ext}_A^1(M, M) \cong \mathrm{Ext}_A^1(M, N)$. Summarised:

$$(11.6.1) \quad \begin{array}{ccc} \mathrm{Ext}_A^1(M, N) & \xleftarrow{\cong} & \mathrm{Ext}_A^1(M, M) \\ \uparrow \cong & & \\ \mathrm{Ext}_B^1(N, N) & \xrightarrow{\bar{\pi}^*} & \mathrm{Ext}_B^1(\bar{M}, N) \end{array}$$

\square

The technique used to prove Theorem 11.1 also gives the following result.

Theorem 11.7. *Let A and T° be henselian and noetherian local rings and $q : A \rightarrow T^\circ$ a local and flat ring homomorphism with $T^\circ/\mathfrak{m}_{T^\circ} = k$ and $T^\circ \otimes_A k = A$. Suppose $J = (f_1, \dots, f_n)$ is a q -sequence. Put $\bar{T}^\circ = T^\circ/J$, $B = \bar{T}^\circ \otimes_A k$ and let \bar{J} be the image of J in A . Let N be a finite B -module and let M denote the syzygy module $\mathrm{Syz}_n^A N$.*

If $\mathrm{ob}(A/\bar{J}^2 \rightarrow B, N) = 0$ then the natural map $s : \mathrm{Def}_N^{\bar{T}^\circ} \rightarrow \mathrm{Def}_M^{T^\circ}$ is injective.

Proof. We proceed as in the proof of Theorem 9.2 and 11.1. Given deformations ${}^i\mathcal{N}$ of N to \bar{h} for $i = 1, 2$. They map to ${}^i\mathcal{M} := \mathrm{Syz}_n^T({}^i\mathcal{N})$ which we suppose are isomorphic as deformations of M to h . Then the natural syzygy map $s^1 : \mathrm{Ext}_B^1(N, N) \rightarrow \mathrm{Ext}_A^1(M, M)$ induces the map of torsor actions along s of the infinitesimal extensions. The composition of s^1 with $\mathrm{Ext}_A^1(M, M) \rightarrow \mathrm{Ext}_A^1(M, \bar{M}) \cong \mathrm{Ext}_B^1(\bar{M}, \bar{M})$ commutes with the horizontal map in Lemma 11.3 (iii). But Lemma 11.3 (iv) implies that $(u_N)_*$ is injective, hence s^1 is injective too. Proceeding by induction on \mathfrak{m}_S^{n+1} -truncations of the deformations we construct a tower of isomorphisms and conclude by Lemma 6.1. \square

Remark 11.8. Theorem 11.7 resembles [28, Thm. 1]. However Theorem 11.7 makes a sounder statement in a more general setting and has a more transparent proof. Indeed, the various similar results in [28] can be changed and proved accordingly.

12. THE KODAIRA-SPENCER MAP OF COHEN-MACAULAY APPROXIMATIONS

A modular family of objects is roughly speaking a family where the isomorphism class of the fibre changes non-trivially. The Kodaira-Spencer map makes this idea precise. We consider the Kodaira-Spencer classes and maps for families of pairs (algebra, module) and by invoking the long-exact transitivity sequence we relate them to the corresponding notions for the algebra and the module. Then we show that Cohen-Macaulay approximation of modular families under conditions as in Theorem 9.2, 9.4 and 11.1 produce new modular families.

The following is a graded version of [30, II 2.1.5.7].

Definition 12.1. Let $\Lambda \rightarrow S$ and $S \rightarrow \Gamma$ be graded ring homomorphisms with Λ and S concentrated in degree 0. The map $L_{\Gamma/S}^{\text{gr}} \rightarrow L_{S/\Lambda} \otimes_S \Gamma[1]$ in the corresponding distinguished transitivity triangle of (graded) cotangent complexes (see (8.8.1)) is called the *Kodaira-Spencer class* of $\Lambda \rightarrow S \rightarrow \Gamma$.

Composing the Kodaira-Spencer class with the natural augmentation map

$$L_{S/\Lambda} \otimes_S \Gamma[1] \longrightarrow \Omega_{S/\Lambda} \otimes_S \Gamma[1]$$

induces an element $\kappa(\Gamma/S/\Lambda) \in {}_0H^1(S, \Gamma, \Omega_{S/\Lambda} \otimes_S \Gamma)$, the cohomological Kodaira-Spencer class, which is also given as follows. Let $\mathcal{P} = \mathcal{P}_{S/\Lambda}$ denote $S \otimes_{\Lambda} S/I^2$ where I is the kernel of the multiplication map $S \otimes_{\Lambda} S \rightarrow S$. There are two ring homomorphisms j_1 and j_2 from S to \mathcal{P} defined by $j_1 : s \mapsto s \otimes 1$ and $j_2 : s \mapsto 1 \otimes s$. Let $d_{S/\Lambda}$ denote the universal derivation (induced by $j_2 - j_1$). The principal parts of Γ is $\mathcal{P} \otimes_S \Gamma$ (with the j_2 tensor product), which gives an S -algebra extension (via j_1) representing the Kodaira-Spencer class:

$$(12.1.1) \quad \kappa(\Gamma/S/\Lambda) : 0 \rightarrow \Omega_{S/\Lambda} \otimes_S \Gamma \longrightarrow \mathcal{P}_{S/\Lambda} \otimes_S \Gamma \longrightarrow \Gamma \rightarrow 0$$

see [30, III 1.2.6]. Since $\mathcal{P} \otimes_S \Gamma$ has a natural \mathcal{P} -algebra structure, (12.1.1) is also a (graded) algebra lifting of Γ along $\mathcal{P} \rightarrow S$ as in Proposition 8.5. The j_1 -extension $\Gamma \otimes_S \mathcal{P} \rightarrow \Gamma$ is a trivial lifting (split by $\text{id}_{\Gamma} \otimes 1_{\mathcal{P}}$) and the difference in ${}_0H^1(S, \Gamma, \Omega_{S/\Lambda} \otimes_S \Gamma)$ given by Proposition 8.5 (ii) equals $\kappa(\Gamma/S/\Lambda)$, see [30, III 2.1.5]. Moreover, the difference $1_{\mathcal{P}} \otimes \text{id}_{\Gamma} - \text{id}_{\Gamma} \otimes 1_{\mathcal{P}}$ induces $d_{S/\Lambda} \otimes 1_{\Gamma}$ (in degree 0) which is mapped to $\kappa(\Gamma/S/\Lambda)$ by the connecting homomorphism

$$(12.1.2) \quad \partial : \text{Der}_{\Lambda}(S, \Omega_{S/\Lambda} \otimes_S \Gamma) \longrightarrow {}_0H^1(S, \Gamma, \Omega_{S/\Lambda} \otimes_S \Gamma)$$

in the long-exact transitivity sequence, see [30, III 1.2.6.5 and 1.2.7].

In the special case of $\Gamma = T \oplus \mathcal{N}$, \mathcal{N} a T -module and $S = T$, the transitivity sequence of $\Lambda \rightarrow T \rightarrow \Gamma$ is given in Proposition 8.8. The Kodaira-Spencer class equals $\partial(d_{T/\Lambda}) \in \text{Ext}_T^1(\mathcal{N}, \Omega_{T/\Lambda} \otimes_T \mathcal{N})$ and is called the (cohomological) *Atiyah class* and is denoted by $\text{at}_{T/\Lambda}(\mathcal{N})$, cf. [30, IV 2.3.6-7]. The class is represented by the short exact sequence

$$(12.1.3) \quad \text{at}_{T/\Lambda}(\mathcal{N}) : 0 \rightarrow \Omega_{T/\Lambda} \otimes_T \mathcal{N} \longrightarrow \mathcal{P}_{T/\Lambda} \otimes_T \mathcal{N} \longrightarrow \mathcal{N} \rightarrow 0.$$

The *Kodaira-Spencer map* of $\Lambda \rightarrow S \rightarrow \Gamma$

$$(12.1.4) \quad g^{\Gamma} : \text{Der}_{\Lambda}(S) \longrightarrow {}_0H^1(S, \Gamma, \Gamma)$$

is defined by $D \mapsto f_*^D \kappa(\Gamma/S/\Lambda)$ where $f^D : \Omega_{S/\Lambda} \rightarrow S$ corresponds to D . Pushout of (12.1.1) by $f^D \otimes \text{id}_{\Gamma}$ gives the corresponding algebra lifting of Γ along $S[\varepsilon] \rightarrow S$ given by $g^{\Gamma}(D)$.

Proposition 12.2. Let Γ denote the graded S -algebra $T \oplus \mathcal{N}$ where $\Lambda \rightarrow S$ and $S \rightarrow T$ are (ungraded) ring homomorphisms and \mathcal{N} is a T -module. Consider the

transitivity sequence of $S \rightarrow T \xrightarrow{i} \Gamma$ in Proposition 8.8:

$$\dots \rightarrow \operatorname{Der}_S(T, \Omega_{S/\Lambda} \otimes_S T) \xrightarrow{\partial} \operatorname{Ext}_T^1(\mathcal{N}, \Omega_{S/\Lambda} \otimes_S \mathcal{N}) \xrightarrow{u} {}_0\operatorname{H}^1(S, \Gamma, \Omega_{S/\Lambda} \otimes_S \Gamma) \xrightarrow{i^*} \operatorname{H}^1(S, T, \Omega_{S/\Lambda} \otimes_S T) \xrightarrow{\partial} \dots$$

- (i) The map i^* takes the Kodaira-Spencer class $\kappa(\Gamma/S/\Lambda)$ to $\kappa(T/S/\Lambda)$.
- (ii) Assume $\kappa(T/S/\Lambda) = 0$ and choose an S -algebra splitting $\sigma : T \rightarrow \mathcal{P} \otimes_S T$. Then there is a class $\kappa(\sigma, \mathcal{N}) = \kappa(T/S/\Lambda, \sigma, \mathcal{N}) \in \operatorname{Ext}_T^1(\mathcal{N}, \Omega_{S/\Lambda} \otimes_S \mathcal{N})$, natural in the argument, which maps to $\kappa(\Gamma/S/\Lambda)$ by u .
- (iii) Let $D(\sigma) \in \operatorname{Der}_\Lambda(T, \Omega_{S/\Lambda} \otimes_S T)$ be the derivation corresponding to the splitting σ and for each $D_1 \in \operatorname{Der}_\Lambda(S)$ let $X_\sigma(D_1)$ denote $f_*^{D_1} D(\sigma) \in \operatorname{Der}_\Lambda(T)$. Then

$$f_*^{D_1} \kappa(\sigma, \mathcal{N}) = f_*^{X_\sigma(D_1)} \operatorname{at}_{T/\Lambda}(\mathcal{N}) \quad \text{in} \quad \operatorname{Ext}_T^1(\mathcal{N}, \mathcal{N}).$$

Proof. The degree zero part of (12.1.1) gives the image $i^* \kappa(\Gamma/S/\Lambda)$ represented by the algebra extension

$$(12.2.1) \quad \kappa(T/S/\Lambda) : \quad 0 \rightarrow \Omega_{S/\Lambda} \otimes_S T \longrightarrow \mathcal{P}_{S/\Lambda} \otimes_S T \longrightarrow T \rightarrow 0.$$

The degree one part is the short exact sequence of $\mathcal{P} \otimes_S T$ -modules

$$(12.2.2) \quad \tilde{\alpha} = \tilde{\alpha}(T/S/\Lambda, \mathcal{N}) : \quad 0 \rightarrow \Omega_{S/\Lambda} \otimes_S \mathcal{N} \longrightarrow \mathcal{P}_{S/\Lambda} \otimes_S \mathcal{N} \longrightarrow \mathcal{N} \rightarrow 0.$$

The splitting σ makes $\tilde{\alpha}$ to a short exact sequence of T -modules which defines $\kappa(T/S/\Lambda, \sigma, \mathcal{N})$.

For (iii) we have $X_\sigma(D_1) = f_*^{X_\sigma(D_1)} d_{T/\Lambda}$ and the result follows from the commutative diagram

$$(12.2.3) \quad \begin{array}{ccccc} D(\sigma) \in \operatorname{Der}_\Lambda(T, \Omega_{S/\Lambda} \otimes_S T) & \xrightarrow{\hspace{2cm}} & \operatorname{Der}_\Lambda(T, \Omega_{T/\Lambda}) \ni d_{T/\Lambda} & & \\ \downarrow \partial & \swarrow f_*^{D_1} & \operatorname{Der}_\Lambda(T, T) & \nwarrow f_*^{X_\sigma(D_1)} & \downarrow \partial \\ & & \downarrow \partial & & \\ & \swarrow f_*^{D_1} & \operatorname{Ext}_T^1(\mathcal{N}, \mathcal{N}) & \nwarrow f_*^{X_\sigma(D_1)} & \\ \kappa(\sigma, \mathcal{N}) \in \operatorname{Ext}_T^1(\mathcal{N}, \Omega_{S/\Lambda} \otimes_S \mathcal{N}) & \xrightarrow{\hspace{2cm}} & \operatorname{Ext}_T^1(\mathcal{N}, \Omega_{T/\Lambda} \otimes_T \mathcal{N}) \ni \operatorname{at}_{T/\Lambda}(\mathcal{N}) & & \end{array}$$

where the two outer vertical maps are pointed. \square

We call $\kappa(\sigma, \mathcal{N})$ for the Kodaira-Spencer class of $(T/S/\Lambda, \sigma, \mathcal{N})$. Define the *Kodaira-Spencer map of $(T/S/\Lambda, \sigma, \mathcal{N})$*

$$(12.2.4) \quad g^{(\sigma, \mathcal{N})} : \operatorname{Der}_\Lambda(S) \longrightarrow \operatorname{Ext}_T^1(\mathcal{N}, \mathcal{N})$$

by $g^{(\sigma, \mathcal{N})}(D) := (f^D \otimes \operatorname{id})_* \kappa(\sigma, \mathcal{N})$.

In the case $T = S \otimes_\Lambda T^o$ we always choose the S -algebra splitting $S \otimes_\Lambda T^o \rightarrow \mathcal{P}_{S/\Lambda} \otimes_S S \otimes_\Lambda T^o \cong \mathcal{P}_{S/\Lambda} \otimes_\Lambda T^o$ given by $s \otimes t \mapsto j_1(s) \otimes t$. In particular $\kappa(T/S/\Lambda) = 0$ and we get a canonical Kodaira-Spencer class $\kappa(\mathcal{N})$ and a corresponding Kodaira-Spencer map $g^\mathcal{N}$.

Remark 12.3. There is no reason to believe that $\kappa(\sigma, \mathcal{N})$ maps to $\operatorname{at}_{T/\Lambda}(\mathcal{N})$ in diagram (12.2.3) for any choice of σ . While there is a canonical map of short exact sequences (of S -modules)

$$\begin{array}{ccccccc} \kappa(\sigma, \mathcal{N}) : & 0 & \longrightarrow & \Omega_{S/\Lambda} \otimes_S \mathcal{N} & \longrightarrow & \mathcal{P}_{S/\Lambda} \otimes_S \mathcal{N} & \longrightarrow \mathcal{N} \longrightarrow 0 \\ & & & \downarrow & & \downarrow \tau & \parallel \\ \operatorname{at}_{T/\Lambda}(\mathcal{N}) : & 0 & \longrightarrow & \Omega_{T/\Lambda} \otimes_T \mathcal{N} & \longrightarrow & \mathcal{P}_{T/\Lambda} \otimes_T \mathcal{N} & \longrightarrow \mathcal{N} \longrightarrow 0 \end{array}$$

τ is in general not T -linear. However, in the case $S \rightarrow T$ is smooth then $\Omega_{S/\Lambda} \otimes T \rightarrow \Omega_{T/\Lambda} \rightarrow \Omega_{T/S}$ is split exact. There is a non-canonical lifting of the universal derivation in $\text{Der}_\Lambda(T, \Omega_{T/\Lambda})$ to $\text{Der}_\Lambda(T, \Omega_{S/\Lambda} \otimes T)$ and the corresponding choice of splitting σ makes τ T -linear and so (or by (12.2.3)) $\kappa(\sigma, \mathcal{N})$ maps to $\text{at}_{T/\Lambda}(\mathcal{N})$.

Example 12.4. Another special case is given by the base change of $h : S \rightarrow T$ with itself to $h \otimes T : T \rightarrow T \otimes_S T = T^{\otimes 2}$ and a T -flat $T^{\otimes 2}$ -module \mathcal{N} , cf. Section 7. Then $\kappa(T^{\otimes 2}/T/S, \mathcal{N})$ in $\text{Ext}_{T^{\otimes 2}}^1(\mathcal{N}, \Omega_{T^{\otimes 2}/T} \otimes \mathcal{N})$ equals $\text{at}_{T^{\otimes 2}/T}(\mathcal{N})$. The multiplication map $\mu_{T^{\otimes 2}/T} : \mathcal{P}_{T^{\otimes 2}/T} \rightarrow T^{\otimes 2}$ equals $\text{id}_T \otimes \mu_{T/S} : T \otimes_S \mathcal{P}_{T/S} \rightarrow T^{\otimes 2}$. It follows that $\text{at}_{T^{\otimes 2}/T}(\mathcal{N})$ maps to $\text{at}_{T/S}(\mathcal{N})$ in $\text{Ext}_T^1(\mathcal{N}, \Omega_{T/S} \otimes \mathcal{N})$ by the natural map. If $\mathcal{N} = T$ then $\text{at}_{T/S}(T) = 0$, but in general $\text{at}_{T^{\otimes 2}/T}(T) \neq 0$.

Example 12.5. The transitivity sequence of $\Lambda \rightarrow T \xrightarrow{i} \Gamma$ and J with $\Gamma = T \oplus \mathcal{N}$ and $J = J_0 \oplus J_1$ in Proposition 8.8

$$(12.5.1) \quad 0 \rightarrow \text{Hom}_T(\mathcal{N}, J_1) \rightarrow {}_0\text{Der}_\Lambda(\Gamma, J) \xrightarrow{i^*} \text{Der}_\Lambda(T, J_0) \rightarrow \text{Ext}_T^1(\mathcal{N}, J_1) \rightarrow \dots$$

suggests the following characterisation. An element $\mathcal{D} \in {}_0\text{Der}_\Lambda(\Gamma, J)$ is given by its degree 0 restriction $D := i^*(\mathcal{D}) \in \text{Der}_\Lambda(T, J_0)$ and its degree 1 restriction $\nabla_D := \mathcal{D}|_{\mathcal{N}} \in \text{Hom}_\Lambda(\mathcal{N}, J_1)$ which should satisfy the following Leibniz rule: For all t in T and n in \mathcal{N}

$$(12.5.2) \quad \nabla_D(tn) = t\nabla_D(n) + D(t)n.$$

With notation as in Proposition 12.2 recall that $\kappa(\Gamma/S/\Lambda) = \partial(d_{S/\Lambda} \otimes 1_T)$ in the transitivity sequence of $\Lambda \rightarrow S \rightarrow \Gamma$:

$$(12.5.3) \quad 0 \rightarrow {}_0\text{Der}_S(\Gamma, \Omega_{S/\Lambda} \otimes \Gamma) \rightarrow {}_0\text{Der}_\Lambda(\Gamma, \Omega_{S/\Lambda} \otimes \Gamma) \rightarrow \text{Der}_\Lambda(S, \Omega_{S/\Lambda} \otimes \Gamma) \xrightarrow{\partial} {}_0\text{H}^1(S, \Gamma, \Omega_{S/\Lambda} \otimes \Gamma) \rightarrow \dots$$

Hence $\kappa(\Gamma/S/\Lambda) = 0$ if and only if there exists a $D \in \text{Der}_\Lambda(T, \Omega_{S/\Lambda} \otimes T)$ which restricts to $d_{S/\Lambda} \otimes 1_T$ and a $\nabla_D \in \text{Hom}_\Lambda(\mathcal{N}, \Omega_{S/\Lambda} \otimes \mathcal{N})$ satisfying (12.5.2). As a well known special case ($S = T$) we get $\text{at}_{T/\Lambda}(\mathcal{N}) = 0$ if and only if there exists a $\nabla \in \text{Hom}_\Lambda(\mathcal{N}, \Omega_{T/\Lambda} \otimes \mathcal{N})$ satisfying (12.5.2) with $D = d_{T/\Lambda} \in \text{Der}_\Lambda(T, \Omega_{T/\Lambda})$ (i.e. ∇ is a connection), or equivalently, a graded derivation $\mathcal{D} \in {}_0\text{Der}_\Lambda(\Gamma, \Omega_{T/\Lambda} \otimes \Gamma)$ restricting to $d_{T/\Lambda}$. Note that (12.5.1) with $J = \Omega_{T/\Lambda} \otimes \Gamma$ equals (12.5.3) in this case.

Recall the maps of cohomology groups $\sigma_j^1(I)$ and $\tau_j^1(I)$ in (8.9.1) and (8.9.2).

Proposition 12.6. *In addition to the assumptions in Lemma 8.9 suppose $\Lambda \rightarrow S$ is a ring homomorphism. For $j = 1, 2$ the following holds:*

- (i) *The map $\sigma_j^1(\Omega_{S/\Lambda})$ takes $\kappa(\Gamma_0/S/\Lambda)$ to $\kappa(\Gamma_j/S/\Lambda)$ and the Kodaira-Spencer maps $g^{\Gamma_i} : \text{Der}_\Lambda(S) \rightarrow {}_0\text{H}^1(S, \Gamma_i, \Gamma_i)$ commute with σ_j^1 , i.e. $\sigma_j^1 g^{\Gamma_0} = g^{\Gamma_j}$.*
- (ii) *Assume $\kappa(T/S/\Lambda) = 0$ and choose an S -algebra splitting $\sigma : T \rightarrow \mathcal{P} \otimes_S T$. Then $\tau_j^1(\Omega_{S/\Lambda})$ maps $\kappa(\sigma, \mathcal{N})$ to $\kappa(\sigma, X_j)$ and the Kodaira-Spencer maps $g^{(\sigma, X_i)} : \text{Der}_\Lambda(S) \rightarrow \text{Ext}_T^1(X_i, X_i)$ commute with τ_j^1 , i.e. $\tau_j^1 g^{(\sigma, \mathcal{N})} = g^{(\sigma, X_j)}$.*

Proof. (i): Put $\kappa_j = \kappa(\Gamma_j/S/\Lambda)$, $\Omega = \Omega_{S/\Lambda}$ and let $\Gamma(\iota) : \Gamma_0 \rightarrow \Gamma_2$ denote the graded ring homomorphism induced from ι . Then $\Gamma(\iota)$ induces a map of short exact sequences $\kappa_0 \rightarrow \kappa_2$, hence a map of short exact sequences $\Gamma(\iota)_* \kappa_0 \rightarrow \Gamma(\iota)^* \kappa_2$, i.e. $\sigma_2(\Omega)(\kappa_0) = (\Gamma(\iota)^*)^{-1} \Gamma(\iota)_* \kappa_0 = \kappa_2$. The maps $\sigma_2(\Omega)$ and $\sigma_2(S)$ commute with the covariant action of $\text{Der}_\Lambda(S)$, hence the second assertion follows from the first. The arguments for the cases $j = 1$ and (ii) are similar. \square

There are corresponding *local* Kodaira-Spencer maps given as follows. Let $t \in \text{Spec } T$ map to $s \in \text{Spec } S$ and consider the localisations $S_{\mathfrak{p}_s} \rightarrow T_{\mathfrak{p}_t}$ and $S_{\mathfrak{p}_s} \rightarrow \Gamma_{\mathfrak{p}_t}$

and the induced map $\Lambda \rightarrow S_{\mathfrak{p}_s}$. The localisation map ${}_0H^1(S, \Gamma, \Omega_{S/\Lambda} \otimes_S \Gamma) \rightarrow {}_0H^1(S_{\mathfrak{p}_s}, \Gamma_{\mathfrak{p}_t}, \Omega_{S_{\mathfrak{p}_s}/\Lambda} \otimes_{S_{\mathfrak{p}_s}} \Gamma_{\mathfrak{p}_t})$ maps $\kappa(\Gamma/S/\Lambda)$ to $\kappa(\Gamma_{\mathfrak{p}_t}/S_{\mathfrak{p}_s}/\Lambda)$. Let $\bar{\kappa}(\Gamma_{\mathfrak{p}_t}/S_{\mathfrak{p}_s}/\Lambda)$ denote the image in ${}_0H^1(S_{\mathfrak{p}_s}, \Gamma_{\mathfrak{p}_t}, \Omega_{S_{\mathfrak{p}_s}/\Lambda} \otimes_{S_{\mathfrak{p}_s}} \Gamma(t))$ by the map induced from $\Gamma_{\mathfrak{p}_t} \rightarrow \Gamma_{\mathfrak{p}_t} \otimes_{S_{\mathfrak{p}_s}} k(s) = \Gamma(t)$. Assume that $\Gamma_{\mathfrak{p}_t}$ is $S_{\mathfrak{p}_s}$ -flat. Then the natural base change map

$$(12.6.1) \quad {}_0H^1(k(s), \Gamma(t), \Omega_{S_{\mathfrak{p}_s}/\Lambda} \otimes_{S_{\mathfrak{p}_s}} \Gamma(t)) \longrightarrow {}_0H^1(S_{\mathfrak{p}_s}, \Gamma_{\mathfrak{p}_t}, \Omega_{S_{\mathfrak{p}_s}/\Lambda} \otimes_{S_{\mathfrak{p}_s}} \Gamma(t))$$

is an isomorphism, see [30, II 2.2]. With this identification we define $g^{\Gamma(t)}(D) := (f^D \otimes \text{id}_{\Gamma(t)})_* \bar{\kappa}(\Gamma_{\mathfrak{p}_t}/S_{\mathfrak{p}_s}/\Lambda)$ for any D in $\text{Der}_{\Lambda}(S_{\mathfrak{p}_s}, k(s))$ and obtain local Kodaira-Spencer maps at t of Γ , and (similarly) of T , respectively:

$$(12.6.2) \quad g^{\Gamma(t)} : \text{Der}_{\Lambda}(S_{\mathfrak{p}_s}, k(s)) \longrightarrow {}_0H^1(k(s), \Gamma(t), \Gamma(t))$$

$$(12.6.3) \quad g^T(t) : \text{Der}_{\Lambda}(S_{\mathfrak{p}_s}, k(s)) \longrightarrow H^1(k(s), T(t), T(t))$$

commuting with the natural map ${}_0H^1(k(s), \Gamma(t), \Gamma(t)) \rightarrow H^1(k(s), T(t), T(t))$ in Proposition 8.8. Note that if Λ is an algebraically closed field then $\text{Der}_{\Lambda}(S_{\mathfrak{p}_s}, k(s))$ is canonically isomorphic to the Zariski tangent space at any closed point $s \in \text{Spec } S$.

Let \mathcal{P} and Ω denote $\mathcal{P}_{S_{\mathfrak{p}_s}/\Lambda}$ and $\Omega_{S_{\mathfrak{p}_s}/\Lambda}$ respectively. Assume $\Gamma = T \oplus \mathcal{N}$. As in the global case we get a graded algebra extension representing $\bar{\kappa}(\Gamma_{\mathfrak{p}_t}/S_{\mathfrak{p}_s}/\Lambda)$ which in degree 1 is a short exact sequence $\tilde{\alpha}(t) : \Omega \otimes \mathcal{N}(t) \rightarrow k(s) \otimes \mathcal{P} \otimes \mathcal{N}_{\mathfrak{p}_t} \rightarrow \mathcal{N}(t)$. If $\bar{\kappa}(\Gamma_{\mathfrak{p}_t}/S_{\mathfrak{p}_s}/\Lambda) = 0$ we choose an S -algebra splitting $\sigma : T(t) \rightarrow k(s) \otimes \mathcal{P} \otimes \Gamma_{\mathfrak{p}_t}$ and the local Kodaira-Spencer class $\bar{\kappa}(\sigma, \mathcal{N}_{\mathfrak{p}_t})$ in $\text{Ext}_{T(t)}^1(\mathcal{N}(t), \Omega \otimes \mathcal{N}(t))$ is represented by the obtained short exact sequence of $T(t)$ -modules. Then we define the local Kodaira-Spencer map of $(\Gamma_{\mathfrak{p}_t}/S_{\mathfrak{p}_s}/\Lambda, \sigma, \mathcal{N}_{\mathfrak{p}_t})$

$$(12.6.4) \quad g^{(\sigma, \mathcal{N})}(t) : \text{Der}_{\Lambda}(S_{\mathfrak{p}_s}, k(s)) \longrightarrow \text{Ext}_{T(t)}^1(\mathcal{N}(t), \mathcal{N}(t))$$

by $g^{(\sigma, \mathcal{N})}(t)(D) := (f^D \otimes \text{id})_* \bar{\kappa}(\sigma, \mathcal{N}_{\mathfrak{p}_t})$.

Similarly, the class $g^{\Gamma(t)}(D)$ is represented by a lifting of graded algebras $\Gamma' \rightarrow \Gamma(t)$ along $k(s)[\varepsilon] \rightarrow k(s)$. If the lifting $g^T(t)(D) : T' \rightarrow T(t)$ splits, a choice of splitting makes the short exact sequence $\tilde{\alpha}(t)(D) : \varepsilon \mathcal{N}(t) \rightarrow N' \rightarrow \mathcal{N}(t)$ T -linear and defines $\tilde{\alpha}(t)(D)$ as an extension in the subspace $\text{Ext}_{T(t)}^1(\mathcal{N}(t), \mathcal{N}(t))$ of $\text{Ext}_{T'}^1(\mathcal{N}(t), \mathcal{N}(t))$.

We assume that Λ is an algebraically closed field k for the rest of this section.

Definition 12.7. Let $h : S \rightarrow T$ be a local flat map of noetherian k -algebras and \mathcal{N} an S -flat T -module. Put $\Gamma = T \oplus \mathcal{N}$, $A = T \otimes_S k$, $N = \mathcal{N} \otimes_S k$ and $\Gamma(0) = \Gamma \otimes_S k = A \oplus N$. We say that (h, \mathcal{N}) is *locally modular* if the local Kodaira-Spencer map $g^{\Gamma(0)} : \text{Der}_k(S, k) \rightarrow {}_0H^1(k, \Gamma(0), \Gamma(0))$ is injective. If in addition $T = S \hat{\otimes}_k A$ then \mathcal{N} is locally modular if $g^{\mathcal{N}(0)} : \text{Der}_k(S, k) \rightarrow \text{Ext}_A^1(N, N)$ is injective.

Let $h^{\text{ft}} : S \rightarrow T$ be a faithfully flat finite type map of noetherian k -algebras with a k -point $t \in \text{Spec } T$ mapping to $s \in \text{Spec } S$ and \mathcal{N} is an S -flat finite T -module. We say that $(h^{\text{ft}}, \mathcal{N})$ is *modular at t* if the henselisation of $(h^{\text{ft}}, \mathcal{N})$ at t is locally modular. If A is a finite type k -algebra and $T = S \otimes_k A$, then \mathcal{N} is modular at t as T -module if its henselisation at t is locally modular. Let $\nabla(h, \mathcal{N})$ ($\nabla_T(\mathcal{N})$ if $T = S \otimes_k A$) denote the set of k -points $t \in \text{Supp } \mathcal{N}$ where (h, \mathcal{N}) (respectively \mathcal{N} as T -module) is modular.

Corollary 12.8. Let $h : S \rightarrow T$ be a finite type Cohen-Macaulay map of k -algebras and let \mathcal{N} be in mod_h^{ft} . Let $\mathcal{N} \xrightarrow{\iota} \mathcal{L}' \rightarrow \mathcal{M}'$ and $\mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N}$ be a $\hat{\text{D}}_h^{\text{ft}}$ -hull and a MCM_h -approximation for \mathcal{N} respectively.

- (i) Suppose $\text{Hom}_{T(t)}(\mathcal{N}(t), \mathcal{M}'(t)) = 0$ for all $t \in \mathfrak{m}\text{-Spec } T$ (e.g. there is an h -regular element contained in $\text{Ann}_T \mathcal{N}$). Then

$$\nabla(h, \mathcal{N}) = \nabla(h, \mathcal{L}') \quad \text{and} \quad \nabla_T(\mathcal{N}) = \nabla_T(\mathcal{L}') \quad \text{if } T = S \otimes_k A.$$

(ii) Suppose $\text{Hom}_{T(t)}(\mathcal{L}(t), \mathcal{N}(t)) = 0$ for all $t \in \mathbf{m}\text{-Spec } T$. Then

$$\nabla(h, \mathcal{N}) = \nabla(h, \mathcal{M}) \quad \text{and} \quad \nabla_T(\mathcal{N}) = \nabla_T(\mathcal{M}) \text{ if } T = S \otimes_k A.$$

Proof. (i) Let t be a k -point in $\text{Spec } T$. By Theorem 5.1 $\iota(t)$ is a $\hat{D}_{T(t)}$ -hull for $\mathcal{N}(t)$ and by Proposition 6.2 $\iota(t)$ is minimal if and only if $\iota_{\mathfrak{p}_t}$ is minimal. In particular; the minimal hull of $\mathcal{N}_{\mathfrak{p}_t}$ is a direct summand of $\mathcal{L}'_{\mathfrak{p}_t}$. We therefore assume that $\iota_{\mathfrak{p}_t}$ and hence $\iota(t)$ is minimal.

By the condition, the map τ_2^1 in (8.9.2) is injective. It implies that the map $\sigma_2^1 : {}_0H^1(k(s), \Gamma_0(t), \Gamma_0(t)) \rightarrow {}_0H^1(k(s), \Gamma_2(t), \Gamma_2(t))$ in (8.9.1) is injective and Proposition 12.6 gives the statement. (ii) is similar. \square

Example 12.9. Let A be a CM finite type k -algebra and domain of dimension ≥ 2 . Let $h : A \rightarrow T = A^{\otimes 2}$ be the base change by $S = A$ and $\mathcal{N} = A$ be the A -flat T -module defined by the multiplication map $T \rightarrow A$. Let $\Delta \subseteq \text{Spec } T$ denote the closed points on the diagonal and let t be a closed point in $\text{Spec } T$ mapping to s in $\text{Spec } A$. If $t \notin \Delta$ then $\mathcal{N}(t) = 0$. If $t \in \Delta$ then $T(t) \cong A_{\mathfrak{p}_s}$ and $\mathcal{N}(t) \cong k(s)$. The local Kodaira-Spencer class $\bar{\kappa}(\mathcal{N}(t)) \in \text{Ext}_{A_{\mathfrak{p}_s}}^1(k(s), \Omega_{A_{\mathfrak{p}_s}/k} \otimes k(s))$ is represented by

$$(12.9.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{A_{\mathfrak{m}}/k} \otimes k(s) & \longrightarrow & k(s) \otimes \mathcal{P}_{A_{\mathfrak{m}}/k} & \longrightarrow & k(s) \longrightarrow 0 \\ & & \cong \uparrow \delta & & \cong \uparrow \chi & & \parallel \\ 0 & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 & \longrightarrow & A/\mathfrak{m}^2 & \longrightarrow & k(s) \longrightarrow 0 \end{array}$$

(with $\mathfrak{m} = \mathfrak{p}_s$) where $\delta(\bar{x}) = d_{A_{\mathfrak{m}}/k}(x) \otimes 1$ and χ is induced by $1 \otimes j_2$ (note that if $x, y \in \mathfrak{m}$ then $1 \otimes j_2(xy) = 1 \otimes ([j_2(x) - j_1(x)][j_2(y) - j_1(y)]) \in k(s) \otimes I^2$). The local Kodaira-Spencer map $g^{\mathcal{N}}(t)$ is given by the pushout

$$(12.9.2) \quad \varphi \in \text{Hom}_{k(s)}(\mathfrak{m}/\mathfrak{m}^2, k(s)) \longrightarrow \text{Ext}_{A_{\mathfrak{m}}}^1(k(s), k(s)) \ni \varphi_* \bar{\kappa}(\mathcal{N}(t))$$

which is an isomorphism. By Corollary 12.8 we have $\nabla_T(\mathcal{N}) = \nabla_T(\mathcal{L}') = \Delta$. Put $\mathcal{Q}' = \text{Hom}_T(\omega_h, \mathcal{L}')$. By Proposition 9.7 also the local Kodaira-Spencer map $g^{\mathcal{Q}'}(t)$ is injective for $t \in \Delta$. Hence $\nabla_T(\mathcal{Q}') = \Delta$. Note that $\mathcal{L}'(t)$ and $\mathcal{Q}'(t)$ are rigid for $t \notin \Delta$.

Corollary 12.10. Suppose A is a finite type Cohen-Macaulay k -algebra and S a noetherian k -algebra and put $h : S \rightarrow T = S \otimes_k A$. Let $J = (f_1, \dots, f_n)$ be an A -sequence, put $B = A/J$ and let $\bar{h} : S \rightarrow \bar{T} = S \otimes_k B$ be the induced Cohen-Macaulay map. Suppose \mathcal{N} is in $\text{MCM}_{\bar{h}} \subseteq \text{mod}_{\bar{h}}^{\text{fl}}$ and let $\mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N}$ be an $\text{MCM}_{\bar{h}}$ -approximation of \mathcal{N} . Assume $\text{ob}(T/(JT)^2 \rightarrow \bar{T}, \mathcal{N}) = 0$. Then

$$\nabla_{\bar{T}}(\mathcal{N}) = \nabla_T(\mathcal{M}) \cap \text{Supp } \bar{T}.$$

Proof. By Proposition 8.7 there is a lifting $\mathcal{N}_1 \rightarrow \mathcal{N}$ of \mathcal{N} to $T_1 = T/(JT)^2$. It induces liftings $\mathcal{N}_1(t) \rightarrow \mathcal{N}(t)$ for all k -points t in $\text{Supp } \bar{T}$. The inclusion $\bar{\tau} : \text{Ext}_{\bar{T}(t)}^1(\mathcal{N}(t), \mathcal{N}(t)) \rightarrow \text{Ext}_{T(t)}^1(\mathcal{M}(t), \mathcal{M}(t))$ in (11.6.1) commutes with the local Kodaira-Spencer maps. Proceed as in the proof of Corollary 12.8. \square

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